FACULTY OF SCIENCE

MASTER PROGRAM OF PURE MATHEMATICS

DYNAMICS AND BIFURCATION OF HIGHER ORDER RATIONAL DIFFERENCE EQUATIONS

By
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Supervisor
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Thesis
July 2, 2017

BIRZEIT UNIVERSITY

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1. Contents

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2. Abstract

The main goal of this Thesis is to study Neimark-Sacker bifurcation of higher order rational difference equations.
We study the third order equations

$$X_{n+1} = \frac{\beta X_n + X_{n-2}}{A + X_{n-1}}$$

with positive parameters and non-negative initial conditions. Moreover, we give details of dynamic behavior and direction of Neimark-Sacker bifurcation.
Also, we study the fourth order equation

$$X_{n+1} = \frac{\beta X_n + X_{n-3}}{A + X_{n-1}}$$

with positive parameters and non-negative initial conditions. We give details of dynamic behavior and direction of Neimark-Sacker bifurcation.
Finally we give some numerical results that show the solution, the dynamical behavior of each equation, and the phase portrait at the bifurcation value.

3. Introduction

In this chapter we mainly introduce the normal form theorem and proof mentioned in Kuznetsov's book, [1].
In practical applications that involve difference equations it very often happens that the difference equation contains parameters and the value of these parameters are often only known approximately. In particular they are generally determined by measurements which are not exact. For that reason it is important to study the behavior of solutions and examine their dependence on the parameters. This study leads to the area referred to as bifurcation theory. The term bifurcation refers to the phenomenon of a system exhibiting new dynamical behavior as the parameter is varied. It can happen that a slight variation in a parameter can have significant impact on the solution. Bifurcation theory is a very deep and complicated area involving lots of current research.

Definition 3.0.1. [5] A point $X^* = (x^*, x^*, \cdots x^*)$ is said to be a fixed point of the map
$x_{n+1} = f(x_n, x_{n-1}, \cdots x)$ if $f(x^*, x^*, \cdots, x^*) = X^*$

Definition 3.0.2. [7] Consider the non-linear difference equation

$$X_{n+1} = AX_n + F(X_n)$$

Where A is $k \times k$ matrix, $X_n \in R^k$ for every $n > 0$, $F \in C[R^k, R^k]$.
Then the following statements hold,

1. If all the eigenvalues of A lie in the open unit disk $|\lambda| < 1$, then the fixed point and consequently the previous equation is asymptotically stable

2. If at least one of the eigenvalues of A has absolute value greater than one, then the fixed point and consequently the previous equation is unstable

3. If all the eigenvalues of A lie in the closed unit disk $|\lambda| \leq 1$, and at least one eigenvalue of A has absolute value equal one, then the stability can't be determined.

As the parameters vary, the phase portrait also varies. There are two possibilities: either the system remains topologically equivalent to the original one, or its topology changes.

Definition 3.0.3. [1] The appearance of topologically non equivalent phase portrait under variation of parameter is called bifurcation.

There are several types of bifurcation, the saddle-node bifurcation, period-doubling bifurcation, Neimark-Sacker bifurcation.

Definition 3.0.4. [1] The bifurcation associated with the appearance of an eigenvalue $\mu = 1$ is called fold or (tangent) bifurcation.

This bifurcation is also referred to as a limit point, saddle-node bifurcation, turning point, among others.

Definition 3.0.5. [1] The bifurcation associated with the appearance of an eigenvalue $\mu = -1$ is called flip or (period-doubling) bifurcation.

Definition 3.0.6. [1] The bifurcation corresponding to the presence of two eigenvalues $\lambda_{1,2} = e^{\pm i\theta_0}$, $0 < \theta_0 < \pi$, is called a Neimark-Sacker (or torus) bifurcation.

The fold and flip bifurcations are possible if $n \geq 1$, but for the Neimark-Sacker bifurcation we need $n \geq 2$.

Example 3.0.1. {*The fold bifurcation*}
Consider the second order difference equation,

$$X_{n+1} = \frac{\beta X_n + X_{n-1}}{\beta + 1 + X_{n-1}} \tag{3.0.1}$$

with a positive parameter β, and a unique fixed point $X^* = (0, 0)$.
Let $\begin{pmatrix} U_n \\ W_n \end{pmatrix} = \begin{pmatrix} X_n \\ X_{n-1} \end{pmatrix}$

Then we have $\begin{pmatrix} U_{n+1} \\ \\ W_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\beta U_n + W_n}{\beta+1+W_n} \\ \\ U_n \end{pmatrix}$

The Jacobian matrix represents the previous matrix is

$$J = \begin{pmatrix} \frac{\beta}{\beta+1} & \frac{1}{\beta+1} \\ 1 & 0 \end{pmatrix}$$

The characteristic equation is $P(\lambda) = |J - \lambda I| = 0$.

$P(\lambda) = \lambda^2 - \frac{\beta}{\beta+1}\lambda - \frac{1}{\beta+1}$.

Solving $\lambda^2 - \frac{\beta}{\beta+1}\lambda - \frac{1}{\beta+1} = 0$ we get,

$\lambda = \dfrac{\beta \pm (\beta+2)}{2(\beta+1)} = \begin{cases} 1 \\ \frac{-1}{\beta+1} \end{cases}$

So there exists an eigenvalue $\lambda = 1$. Note that $\lambda_2 < 1$ for every β.

Example 3.0.2. {*Flip bifurcation*}
Consider the following logistic map $f_\alpha(X) = \alpha X(1-X)$.

This map has a unique fixed point $X^* = 1 - \dfrac{1}{\alpha}$.

The eigenvalue is $\mu = f_x(\alpha, X^*) = -\alpha X^* + \alpha(1 - X^*) = 2 - \alpha$.

$|2 - \alpha| < 1$ so for $1 < \alpha < 3$ the positive fixed point is stable.

For $\alpha = 3$ then $\mu = -1$, and for $\alpha > 3$ the fixed point X^* becomes unstable.
Notice the existence of period two at $\alpha = 3$.

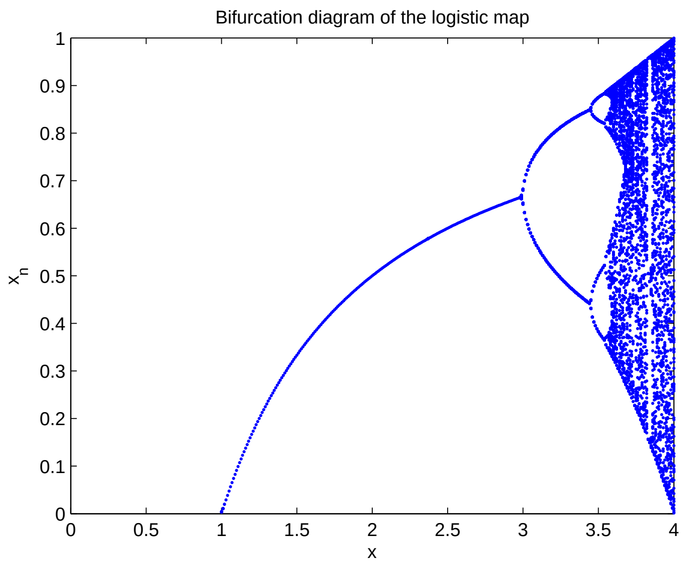

Fig. 3.1: bifurcation diagram of the logistic map

We will study in detail Neimark-Sacker bifurcation.
Neimark-Sacker bifurcation is the birth of a closed invariant curve from a fixed point in dynamical systems with discrete time (iterated maps), when the fixed point changes stability via a pair of complex eigenvalues with unit modulus. The bifurcation can be supercritical or subcritical, resulting in a stable or unstable (within an invariant two-dimensional manifold) closed invariant curve, respectively. When it happens in the *Poincarè* map of a limit cycle, the bifurcation generates an invariant two-dimensional torus in the corresponding ODE.
The Neimark-Sacker bifurcation (NSB) is the equivalent of the Hopf bifurcation for maps. For instance, in the case of a supercritical NSB, a stable focus loses its stability as a parameter is varied with the consequent birth of a stable cycle or quasi-cycle - we'll refer to either of these as a closed invariant curve. In the case of a subcritical NSB, a stable focus enclosed by an unstable closed curve loses its stability with the consequent disappearance of the closed invariant curve as a parameter is varied.

3.1 The normal form of the Neimark-Sacker bifurcation

Consider the following two-dimensional discrete-time system depending on one parameter

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to (1+\alpha)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} +$$

$$(x_1^2+x_2^2)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

where α is the parameter, $\theta = \theta(\alpha)$, $a = a(\alpha)$ and $b = b(\alpha)$ are smooth functions, and $0 < \theta(0) < \pi$, $a(0) \neq 0$.
This system has the fixed point $x_1 = x_2 = 0$ for all α with Jacobian matrix

$$A = (1+\alpha)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The matrix has eigenvalues $\lambda_{1,2} = (1+\alpha)e^{\pm i\theta}$, which makes the previous map invertible near the origin for all small $|\alpha|$. As can be seen, the fixed point at the origin is non-hyperbolic at $\alpha = 0$ due to a complex-conjugate pair of the eigenvalues on the unit circle. To analyze the corresponding bifurcation, introduce the complex variable

$$z = x_1 + ix_2, \quad \bar{z} = x_1 - ix_2, \quad |z|^2 = z\bar{z} = x_1^2 + x_2^2$$

and set $d = a + ib$, the equation for z leads

$$z \to e^{i\theta} z(1+\alpha+d|z|^2) = \mu z + cz|z|^2$$

where $\mu = \mu(\alpha) = (1+\alpha)e^{i\theta(\alpha)}$ and $c = c(\alpha) = e^{i\theta(\alpha)}d(\alpha)$ are complex functions of the parameter α. Using the representation $z = \rho e^{i\varphi}$, we obtain for $\rho = |z|$

$$\rho \to \rho|1+\alpha+d(\alpha)\rho^2|$$

Since

$$|1+\alpha+d(\alpha)\rho^2| = (1+\alpha)\left(1+\frac{2a(\alpha)}{1+\alpha}\rho^2+\frac{|d(\alpha)|^2}{(1+\alpha)^2}\rho^4\right)^{1/2}$$

$$= 1+\alpha+a(\alpha)\rho^2+O(\rho^3).$$

We obtain the following polar form

$$\begin{cases} \rho \to \rho(1+\alpha+a(\alpha)\rho^2+\rho^4 R_\alpha(\rho)) \\ \varphi \to \varphi+\theta(\alpha)+\rho^2 Q_\alpha(\rho) \end{cases}$$

For functions R and Q, which are smooth functions of (ρ, α). Bifurcations of the system's phase portrait as α passes through zero can be analyzed using the latter form, since the mapping for ρ is independent of φ. The equation

$$\rho \to \rho(1+\alpha+a(\alpha)\rho^2+\rho^4 R_\alpha(\rho))$$

defines a one-dimensional dynamical system that has the fixed point $\rho = 0$ for all values of α. The point is linearly stable if $\alpha < 0$; for $\alpha > 0$ the point becomes linearly unstable. The stability of the fixed point at $\alpha = 0$ is determined by the sign of the coefficient $a(0)$. Suppose that $a(0) < 0$; then the origin is (nonlinearly) stable at $\alpha = 0$. Moreover the ρ-map has an additional stable fixed point

$$\rho_0(\alpha) = \sqrt{\frac{-\alpha}{a(\alpha)}} + O(\alpha)$$

for $\alpha > 0$. The φ map describes a rotation by an angle depending on ρ and α; it is approximately equal to $\theta(\alpha)$. Thus, by superposition of the previous mappings, we obtain the bifurcation diagram for the original two-dimensional system .

The system always has a fixed point at the origin. This point is stable for $\alpha < 0$ and unstable for $\alpha > 0$. The invariant curves of the system near the origin look like the orbits near the stable focus of a continuous-time system for $\alpha < 0$ and like orbits near the unstable focus for $\alpha > 0$. At the critical parameter value $\alpha = 0$ the point is nonlinearly stable. The fixed point is surrounded for $\alpha > 0$ by an isolated closed invariant curve that is unique and stable. The curve is a circle of radius $\rho_0(\alpha)$. All orbits starting outside or inside the closed invariant curve, except at the origin, tend to the curve under iterations. This is a Neimark-Sacker bifurcation. This bifurcation can also be presented in (x_1, x_2, α)-space. The appearing family of closed invariant curves, parameterized by α, forms a paraboloid surface.

The case $a(0) > 0$ can be analyzed in the same way. The system undergoes the Neimark-Sacker bifurcation at $\alpha = 0$. Contrary to the considered case, there is an unstable closed invariant curve that disappears when α crosses zero from negative to positive values.

3.2 Generic Neimark-Sacker bifurcation

We now shall prove that any generic two-dimensional system undergoing a Neimark-Sacker bifurcation can be transformed into the form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to (1+\alpha)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} +$$

$$(x_1^2+x_2^2)\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Consider a system

$$x \to f(x,\alpha), \quad x = (x_1, x_2)^T \in \mathbb{R}^2, \alpha \in \mathbb{R}^1.$$

with a smooth function f, which has at $\alpha = 0$ the fixed point $x = 0$ with simple eigenvalues $\lambda_{1,2} = e^{\pm i\theta_0}, 0 < \theta_0 < \pi$. By the Implicit Function Theorem, the system has a unique fixed point $x_0(\alpha)$ in some neighborhood of the origin for all sufficiently small $|\alpha|$, since $\lambda = 1$ is not an eigenvalue of the Jacobian matrix. We can perform a parameter-dependent coordinate shift, placing this fixed point at the origin. Therefore, we may assume without loss of generality that $x = 0$ is the fixed point of the system for $|\alpha|$ sufficiently small. Thus, the system can be written as

$$x \to A(\alpha)x + F(x,\alpha)$$

where F is a smooth vector function whose components $F_{1,2}$ have Taylor expansions in x starting with at least quadratic terms
$F(x,\alpha) = \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + \cdots$, $F(0,\alpha) = 0$ for all sufficiently small $|\alpha|$. The Jacobian matrix $A(\alpha)$ has two eigenvalues

$$\lambda_{1,2}(\alpha) = r(\alpha)e^{\pm i\varphi(\alpha)}$$

where $r(0) = 1, \theta(0) = \theta_0$. Thus, $r(\alpha) = 1 + \beta(\alpha)$ for some smooth function $\beta(\alpha), \beta(0) = 0$. Suppose that $\beta'(0) \neq 0$. Then, we can use β as a new parameter and express the multipliers in terms of $\beta : \lambda_1(\beta) = \lambda(\beta), \lambda_2(\beta) = \bar{\lambda}(\beta)$, where

$$\lambda(\beta) = (1+\beta)e^{i\theta_\beta}$$

with a smooth function $\theta(\beta)$ such that $\theta(0) = \theta_0$.

Lemma 3.2.1. *[1] The map*

$$z \to \lambda z + \frac{g_{20}}{2}z^2 + g_{11}z\bar{z} + \frac{g_{02}}{2}\bar{z}^2 + O(|z^3|)$$

Where $\lambda = \lambda(\beta) = (1+\beta)e^{i\theta_\beta}, g_{ij} = g_{ij}(\beta)$, *can be transformed by an invertible parameter-dependent change of complex coordinate*

$$z = w + \frac{h_{20}}{2}w^2 + h_{11}w\bar{w} + \frac{h_{02}}{2}\bar{w}^2$$

for all sufficiently small $|\beta|$, *into a map without quadratic terms:*

$$w \to \lambda w + O(|w^3|)$$

provided that

$$e^{i\theta_0} \neq 1 \quad e^{i3\theta_0} \neq 1$$

Proof. The inverse change of variables is given by

$$w = z - \frac{h_{20}}{2}z^2 - h_{11}z\bar{z} - \frac{h_{02}}{2}\bar{z}^2 + O(|z^3|)$$

Therefore, in the new coordinate w, the map takes the form

$$\begin{aligned} \tilde{w} = \lambda w &+ \frac{1}{2}(g_{20} + (\lambda - \lambda^2)h_{20})w^2 \\ &+ (g_{11} + (\lambda - |\lambda|^2)h_{11})w\bar{w} \\ &+ \frac{1}{2}(g_{02} + (\lambda - \bar{\lambda}^2)h_{02})\bar{w}^2 \\ &+ O(|w^3|) \end{aligned}$$

Thus, by setting

$$h_{20} = \frac{g_{20}}{\lambda^2 - \lambda}, h_{11} = \frac{g_{11}}{|\lambda|^2 - \lambda}, h_{02} = \frac{g_{02}}{\bar{\lambda}^2 - \lambda}.$$

we kill all the quadratic terms. These substitutions are valid if the denominators are nonzero for all sufficiently small $|\beta|$ including $\beta = 0$. Indeed, this is the case, since

$$\lambda(0)^2 - \lambda(0) = e^{i\theta_0}(e^{i\theta_0} - 1) \neq 0$$
$$|\lambda(0)|^2 - \lambda(0) = 1 - e^{i\theta_0} \neq 0$$
$$\bar{\lambda}(0)^2 - \lambda(0) = e^{i\theta_0}(e^{-i3\theta_0} - 1) \neq 0$$

due to our restrictions on θ_0. □

Assuming that we have removed all quadratic terms, let us try to eliminate the cubic terms as well.

Lemma 3.2.2. *[1] The map*

$$z \to \lambda z + \frac{g_{30}}{6} z^3 + \frac{g_{21}}{2} z^2 \bar{z} + \frac{g_{12}}{2} \bar{z}^2 z + \frac{g_{03}}{6} \bar{z}^3 + O(|z|^4)$$

Where $\lambda = \lambda(\beta) = (1+\beta)e^{i\theta_0}, g_{ij} = g_{ij}(\beta)$ *can be transformed by an invertible parameter-dependent change of coordinates*

$$z = w + \frac{h_{30}}{6} w^3 + \frac{h_{21}}{2} w^2 \bar{w} + \frac{h_{12}}{2} \bar{w}^2 w + \frac{h_{03}}{6} \bar{w}^3$$

for all sufficiently small $|\lambda|$ *, into a map with only one cubic term:*

$$w \to \lambda w + c_1 w^2 \bar{w} + O(|w|^3).$$

provided that

$$e^{2i\theta_0} \neq 1 \quad e^{4i\theta_0} \neq 1$$

Proof. The inverse transformation is

$$w = z - \frac{h_{30}}{6} z^3 - \frac{h_{21}}{2} z^2 \bar{z} - \frac{h_{12}}{2} \bar{z}^2 z - \frac{h_{03}}{6} \bar{z}^3 + O(|z|^4)$$

Therefore,

$$\tilde{w} = \lambda w + \frac{1}{6}\left(g_{30} + (\lambda - \lambda^3) h_{30}\right) w^3 + \frac{1}{2}\left(g_{21} + (\lambda - \lambda|\lambda|^2) h_{21}\right) w^2 \bar{w}$$

$$+\frac{1}{2}\left(g_{12} + (\lambda - \bar{\lambda}|\lambda|^2) h_{12}\right) w \bar{w}^2 + \frac{1}{6}\left(g_{03} + (\lambda - \bar{\lambda}^3) h_{03}\right) \bar{w}^3 + O(|w|^4)$$

Thus, by setting

$$h_{30} = \frac{g_{30}}{\lambda^3 - \lambda}, \quad h_{21} = \frac{g_{21}}{|\lambda|^2 \bar{\lambda} - \lambda}, \quad h_{03} = \frac{g_{03}}{\bar{\lambda}^3 - \lambda}$$

We can annihilate all cubic terms in the resulting map except the $w^2\bar{w}$-term, which must be treated separately. The substitutions are valid since all the involved denominators are nonzero for all sufficiently small $|\beta|$ due to the assumptions concerning θ_0.

One can also try to eliminate the $w^2\bar{w}$-term by formally setting

$$h_{21} = \frac{g_{21}}{\lambda(1 - |\lambda|^2)}$$

This is possible for small $\beta \neq 0$, but the denominator vanishes at $\beta = 0$ for all θ_0. Thus, no extra conditions on θ_0 would help. To obtain a transformation that is smoothly dependent on β, set $h_{21} = 0$, that results in

$$c_1 = \frac{g_{21}}{2}$$

□

Lemma 3.2.3. *[1](Normal form for the Neimark-Sacker bifurcation)*
The map

$$z \to \lambda z + \frac{g_{20}}{2}z^2 + g_{11}z\bar{z} + \frac{g_{02}}{2}\bar{z}^2 + \frac{g_{30}}{6}z^3 + \frac{g_{21}}{2}z^2\bar{z} + \frac{g_{12}}{2}\bar{z}^2 z + \frac{g_{03}}{6}\bar{z}^3 + O(|z|^4)$$

Where

$$\lambda = \lambda(\beta) = (1+\beta)e^{i\theta_\beta}, \qquad g_{ij} = g_{ij}(\beta), \qquad \theta = \theta_0$$

$e^{ik\theta_0} \neq 1$ *, for* $k = 1, 2, 3, 4$. *can be transformed by an invertible parameter dependent change of complex coordinate, which is smoothly dependent on the parameter,*

$$z = w + \frac{h_{20}}{2}w^2 + h_{11}w\bar{w} + \frac{h_{02}}{2}\bar{w}^2 + \frac{h_{30}}{6}w^3 + \frac{h_{21}}{2}w^2\bar{w} + \frac{h_{12}}{2}\bar{w}^2 w + \frac{h_{03}}{6}\bar{w}^3$$

for all sufficiently small $|\beta|$*, into a map with only the resonant cubic term:*

$$w = w\lambda + c_1 w^2\bar{w} + O(|w|^4)$$

where $c_1 = c_1(\beta)$

The truncated superposition of the transformations defined in the two previous lemmas gives the required coordinate change. First, annihilate all the quadratic terms. This will also change the coefficients of the cubic terms. The coefficient of $w^2\bar{w}$ will be $\frac{1}{2}\tilde{g_{21}}$, say, instead of $\frac{1}{2}g_{21}$. Then, eliminate all the cubic terms except the resonant one. The coefficient of this term remains $\frac{1}{2}\tilde{g_{21}}$. Thus, all we need to compute to get the coefficient of c_1 in terms of the given equation is a new coefficient $\frac{1}{2}\tilde{g_{21}}$ of the $w^2\bar{w}$- term after the quadratic transformation. The computations result in the following expression for $c_1(\alpha)$:

$$c_1 = \frac{g_{20}g_{11}(\bar{\lambda} - 3 + 2\lambda)}{2(\lambda^2 - \lambda)(\bar{\lambda} - 1)} + \frac{|g_{11}|^2}{1 - \bar{\lambda}} + \frac{|g_{02}|^2}{2(\lambda^2 - \lambda)} + \frac{g_{21}}{2}$$

which gives, for the critical value of c_1

$$c_1(0) = \frac{g_{20}(0)g_{11}(0)(\bar{\lambda_0} - 3 + 2\lambda_0)}{2(\lambda_0^2 - \lambda_0)(\bar{\lambda_0} - 1)} + \frac{|g_{11}(0)|^2}{1 - \bar{\lambda}_0} + \frac{|g_{02}(0)|^2}{2(\lambda_0^2 - \lambda_0)} + \frac{g_{21}(0)}{2}$$

Where $\lambda_0 = e^{i\theta_0}$

Theorem 3.2.4. *[1]Suppose a two-dimensional discrete-time system*

$$x \to f(x.\alpha). \quad x \in \mathbb{R}^2, \quad \alpha \in \mathbb{R}$$

with smooth f has for all sufficiently small $|\alpha|$, the fixed point $x = 0$ with multipliers

$$\lambda_{1,2}(\alpha) = r(\alpha)e^{\pm i\varphi(\alpha)}$$

Where $r(0) = 0, \varphi(0) = \theta_0$
Let the following conditions be satisfied:

$$r'(0) \neq 0$$

$$e^{ik\theta_0} \neq 1, \quad for \quad k = 1, 2, 3, 4$$

Then, there are smooth invertible coordinate and parameter changes transforming the system into $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \to (1+\beta) \begin{pmatrix} \cos\theta(\beta) & -\sin\theta(\beta) \\ \sin\theta(\beta) & \cos\theta(\beta) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + (y_1^2 + y_2^2) \begin{pmatrix} \cos\theta(\beta) & -\sin\theta(\beta) \\ \sin\theta(\beta) & \cos\theta(\beta) \end{pmatrix} \begin{pmatrix} a(\beta) & -b(\beta) \\ b(\beta) & a(\beta) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + O(|y|^4)$

with $\theta(0) = \theta_0$ and $a(0) = \mathbb{R}e(e^{i\theta_0}c_1(0))$, where $c_1(0)$ is given by

$$c_1(0) = \frac{g_{20}(0)g_{11}(0)(\bar{\lambda_0} - 3 + 2\lambda_0)}{2(\lambda_0^2 - \lambda_0)(\bar{\lambda_0} - 1)} + \frac{|g_{11}(0)|^2}{1 - \bar{\lambda}_0} + \frac{|g_{02}(0)|^2}{2(\lambda_0^2 - \lambda_0)} + \frac{g_{21}(0)}{2}$$

Proof. The only thing left to verify is the formula for $a(0)$. Indeed, by previous Lemmas, the system can be transformed to the complex Poincaré normal form,

$$w \to \lambda(\beta)w + c_1(\beta)w|w|^2 + O(|w|^4)$$

For $\lambda(\beta) = (1 + \beta)e^{i\theta(\beta)}$.
This map can be written as

$$w \to e^{i\theta(\beta)}(1 + \beta + d(\beta)|w|^2)w + O(|w|^4)$$

Where $d(\beta) = a(\beta) + ib(\beta)$ for some real functions $a(\beta), b(\beta)$

$$a(\beta) = \mathbb{R}e(d(\beta)) = \mathbb{R}e(e^{-i\theta(\beta)}c_1(\beta))$$

Thus,

$$a(0) = \mathbb{R}e(e^{-i\theta(0)}c_1(0))$$

□

In Neimark Sacker bifurcation the Jacobian matrix has simple pairs of complex eigenvalues on the unit circle. $\lambda_{1,2} = e^{\pm i\theta_0}$, $0 < \theta_0 < \pi$ and these are the only eigenvalues with $|\lambda| = 1$.
Let $q \in \mathbb{C}^n$ be a complex eigenvector correspond to $\lambda_1 = e^{i\theta_0}$,

$$Aq = e^{i\theta_0}q, \qquad A\bar{q} = e^{-i\theta_0}\bar{q}$$

Introduce also the adjoint eigenvector $p \in \mathbb{C}^n$ having the properties

$$A^T p = e^{-i\theta_0}p, \quad A^T\bar{p} = e^{i\theta_0}\bar{p}$$

and satisfying the normalization property

$$< p, q >= 1$$

Where $< p, q >= \sum_{i=1}^{n} \bar{p}_i q_i$ is the standard scaler product in $\mathbb{C}^n$. The critical real eigenspace T^c corresponding to $\lambda_{1,2}$ is two-dimensional and is spanned by $\{Re(q), Im(q)\}$. The real eigenspace T^{su} corresponding to all eigenvalues of A other than $\lambda_{1,2}$ is $(n-2)$-dimensional. $y \in T^{su}$ if and only if $< p, y >= 0$. Notice that $y \in \mathbb{R}^n$ is real, while $p \in \mathbb{C}^n$ is complex. Therefore, the condition $< p, y >= 0$ implies two real constraints on y. Decompose $x \in \mathbb{R}^n$ as

$$x = zq + \bar{z}\bar{\bar{q}} + y$$

where $z \in \mathbb{C}^1$, and $zq + \bar{z}\bar{q} \in T^c$, $y \in T^{su}$. The complex variable z is a coordinate on T^c. We have

$$\begin{cases} z =< p, x > \\ y = x- < p, x > q- < \bar{p}, x > \bar{q} \end{cases}$$

In these coordinates, the map $\tilde{x} = Ax + F(x), x \in \mathbb{R}^n$ takes the form

$$\begin{cases} \tilde{z} = e^{i\theta_0}z+ < p, F(zq + \bar{z}\bar{q} + y) > \\ \tilde{y} = Ay + F(zq + \bar{z}\bar{q} + y)- < p, F(zq + \bar{z}\bar{q} + y) > q- < \bar{p}, F(zq + \bar{z}\bar{q} + y) > \bar{q} \end{cases}$$

This system is $(n+2)$ dimensional, but we have to remember the two real constraints imposed on y. The system can be written in a form

$$\begin{cases} \widetilde{z} = e^{i\theta_0}z + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2 + \frac{1}{2}G_{21}z^2\bar{z} + < G_{10}, y > z + < G_{01}, y > \bar{z} \\ \widetilde{y} = Jy + \frac{1}{2}H_{20}z^2 + H_{11}z\bar{z} + \frac{1}{2}H_{02}\bar{z}^2 + \frac{1}{2}H_{21}z^2\bar{z} \end{cases}$$

where $G_{20}, G_{11}, G_{02}, G_{21} \in \mathbb{C}^1, G_{01}, G_{10}, H_{ij} \in \mathbb{C}^n$; and the scalar product in $\mathbb{C}^n$ is used.
The complex numbers and vectors can be computed by the formulas

$$\begin{cases} G_{20} =< p, B(q,q) >, G_{11} =< p, B(q,\bar{q}) >, \\ G_{02} =< p, B(\bar{q},\bar{q}) >, G_{21} =< p, C(q,q,\bar{q}) > \end{cases}$$

$$\begin{cases} H_{20} = B(q,q) - < p, B(q,q) > q - < \bar{p}, B(q,q) > \bar{q} \\ H_{11} = B(q,\bar{q}) - < p, B(q,\bar{q}) > q - < \bar{p}, B(q,\bar{q}) > \bar{q} \end{cases}$$

$$\{ < G_{10}, y > =< p, B(q,y) >, < G_{01}, y > =< p, B(\bar{q}, y) >$$

The center manifold in the previous system has the representation

$$Y = V(z,\bar{z}) = \frac{1}{2}w_{20}z^2 + w_{11}z\bar{z} + \frac{1}{2}w_{02}\bar{z}^2$$

where $< q, w_{ij} >= 0$. The vectors $w_{ij} \in \mathbb{C}^n$ can be found from the linear equations

$$\begin{cases} w_{20} = (e^{2i\theta_0}I_3 - J)^{-1}H_{20} \\ w_{11} = (I_3 - J)^{-1}H_{11} \\ w_{02} = (e^{-2i\theta_0}I_3 - J)^{-1}H_{02} \end{cases}$$

These equations have unique solutions. The matrix $(I - A)$ is invertible because 1 is not an eigenvalue of A $(e^{i\theta_0} \neq 1)$ if $e^{3i\theta_0} \neq 1$, the matrices $(e^{\pm 2i\theta_0}I - A)$ are also invertible in C^n because $e^{2i\theta_0}$ are not eigenvalues of A. Thus, generically, the restricted map can be written as

$$\widetilde{z} = e^{i\theta_0}\bar{z} + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2$$

$$+\frac{1}{2}(G_{21} + 2 < p, B(q,(I - J)^{-1}H_{11}) > + < p, B(\bar{q},(e^{2i\theta_0}I - J)^{-1}H_{20}) >)z^2\bar{z}$$

In this generic situation, substituting the value of $G_(ij)$ and $H_(ij)$ and taking into account the identities

$$(I-J)^{-1}q = \frac{1}{1-e^{i\theta_0}}q, \qquad (e^{2i\theta_0}I-J)^{-1}q = \frac{e^{-i\theta_0}}{e^{i\theta_0}-1}q$$

and

$$(I-J)^{-1}\bar{q} = \frac{1}{1-e^{i\theta_0}}\bar{q}, \qquad (e^{2i\theta_0}I-J)^{-1}\bar{q} = \frac{e^{-i\theta_0}}{e^{i\theta_0}-1}\bar{q}$$

$$\tilde{z} = e^{i\theta_0}z + \frac{1}{2}g_{20}z^2 + g_{11}z\bar{z} + \frac{1}{2}g_{02}\bar{z}^2 + \frac{1}{2}g_{21}z^2\bar{z} + \cdots$$

Where

$$g_{20} =< p, B(q,q) >, \qquad g_{11} =< p, B(q,\bar{q}) >, \qquad g_{02} =< p, B(\bar{q},\bar{q}) >$$

$g_{21} =< p, C(q,q,\bar{q}) > +2 < p, B(q,(I-J)^{-1}B(q,\bar{q})) > +$
$< p, B(\bar{q},(e^{2i\theta_0}I-J)^{-1}B(q,q)) > + \cdots$ in the absence of strong resonances, i.e.

$$e^{ik\theta_0} \neq 1, \quad k = 1,2,3,4,$$

So

$$\tilde{z} = e^{i\theta_0}z(1+d(0))|z^2|$$

where the real number $a(0) = \mathbb{R}e(d(0))$, that determines the direction of bifurcation of a closed invariant curve, can be computed by formula

$$\mathbb{R}e(d(0)) = \mathbb{R}e(\frac{e^{-i\theta_0}g_{21}}{2}) - \mathbb{R}e(\frac{(1-2e^{i\theta_0})e^{-2i\theta_0}}{2(1-e^{i\theta_0})}g_{20}g_{11}) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2$$

This compact formula allows us to verify the non-degeneracy of the nonlinear terms at a non-resonant Neimark-Sacker bifurcation of n-dimensional maps with $n \geq 2$. Note that all the computations can be performed in the original basis.

Example 3.2.1. *(Neimark-Sacker bifurcation in the delayed logistic equation)* Consider the following recurrence equation

$$U_{n+1} = rU_n(1-U_{n-1}) \tag{3.2.1}$$

This is a simple population dynamics model, where U_n stands for the density of a population at time n and r is the growth rate. It is assumed that the growth is determined not only by the current population density but also by its density in the past.

Solving $X^* = rX^*(1 - X^*)$

then $X^* = (0, 0)$ or $r(1 - X^*) = 1$, so $X^* = (1 - \frac{1}{r}, 1 - \frac{1}{r})$

Let $\begin{pmatrix} U_n \\ U_{n-1} \end{pmatrix} = \begin{pmatrix} V_n \\ W_n \end{pmatrix}$,then equation (3.2.1) turned to
$\begin{pmatrix} V_{n+1} \\ W_{n+1} \end{pmatrix} = \begin{pmatrix} rV_n(1 - W_n) \\ V_n \end{pmatrix}$

The Jacobian matrix that represents this matrix at the positive fixed point is

$J = \begin{pmatrix} r(1 - X^*) & -rX^* \\ 1 & 0 \end{pmatrix}$
$J = \begin{pmatrix} r(1 - (1 - \frac{1}{r})) & -r(1 - \frac{1}{r}) \\ 1 & 0 \end{pmatrix}$
$J = \begin{pmatrix} 1 & 1 - r \\ 1 & 0 \end{pmatrix}$

The characteristic equation of the Jacobian matrix is

$\begin{vmatrix} 1 - \lambda & 1 - r \\ 1 & -\lambda \end{vmatrix} = 0$

So

$$P(\lambda) = \lambda^2 - \lambda - (1 - r)$$

The roots are

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1 + 4(1 - r)}}{2} = \frac{1 \pm \sqrt{5 - 4r}}{2}$$

For $r > \frac{5}{4}$ there is two complex conjugate roots.

$$|\lambda_1 \lambda_2| = \frac{(1 + \sqrt{5 - 4r})(1 - \sqrt{5 - 4r})}{4} = \frac{1 - (5 - 4r)}{4} = \frac{4(-1 + r)}{4}$$

So at $r = 2$ the positive fixed point loses stability and we have Neimark-

Sacker bifurcation. At $r = 2$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{3}i}{2}$$

So

$$\lambda = e^{i\theta_0}, \theta_0 = \frac{\pi}{3}$$

Also

$$e^{ik\theta_0} \neq 1 \quad for \quad k = 1, 2, 3, 4$$

The eigenvectors of the Jacobian matrix are

$$Jq = e^{i\theta_0}q, \quad J^T p = e^{-i\theta_0}p$$

To find q.

$$\begin{pmatrix} 1 - e^{i\theta_0} & 1 - r \\ 1 & -e^{i\theta_0} \end{pmatrix} q = 0$$

$$\begin{pmatrix} \frac{1-\sqrt{3}i}{2} & -1 \\ 1 & -\frac{1+\sqrt{3}i}{2} \end{pmatrix} q = 0$$

Let $q_2 = 1$ then $\frac{1-\sqrt{3}i}{2}q_1 - 1 = 0$,

$$q_1 = \frac{1}{\frac{1-\sqrt{3}i}{2}} = \frac{1 + \sqrt{3}i}{2}$$

$q = \begin{pmatrix} \frac{1+\sqrt{3}i}{2} \\ 1 \end{pmatrix}$

Solving $J^T P = e^{-i\theta_0} P$

$$\begin{pmatrix} 1 - e^{-i\theta_0} & 1 \\ 1 - r & -e^{-i\theta_0} \end{pmatrix} P = 0$$

$$\begin{pmatrix} \frac{1+\sqrt{3}i}{2} & 1 \\ -1 & \frac{-1+\sqrt{3}i}{2} \end{pmatrix} P = 0$$

Let $P_2 = 1$, then $\frac{1+\sqrt{3}i}{2}P_1 + 1 = 0$

$$P_1 = \frac{-1}{\frac{1+\sqrt{3}i}{2}} = \frac{-1+\sqrt{3}i}{2}$$

$P = \begin{pmatrix} \frac{-1+\sqrt{3}i}{2} \\ 1 \end{pmatrix}$
Normalize P, q. To achieve the normalization $< P, q >= 1$.
$\mu = \bar{P}.q = \begin{pmatrix} \frac{-1-\sqrt{3}i}{2} \\ 1 \end{pmatrix} . \begin{pmatrix} \frac{1+\sqrt{3}i}{2} \\ 1 \end{pmatrix} = \frac{3-\sqrt{3}i}{2}$
So take $q = \begin{pmatrix} \frac{1+\sqrt{3}i}{2} \\ 1 \end{pmatrix}$, $\quad p = \mu^{-1}P = \frac{3-\sqrt{3}i}{2} \begin{pmatrix} \frac{-1+\sqrt{3}i}{2} \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3}i \\ \frac{3-\sqrt{3}i}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}i}{3} \\ \frac{1}{2} - \frac{\sqrt{3}i}{6} \end{pmatrix}$
System (1) can be written as

$$Y_{n+1} = JY_n + G(Y_n) \tag{3.2.2}$$

where $G(Y) = \frac{1}{2}B(Y,Y) + \frac{1}{6}C(Y,Y,Y)$ and $Y_n = \begin{pmatrix} V_n \\ W_n \end{pmatrix}$
$B(Y,Y) = \begin{pmatrix} B_1(Y,Y) \\ 0 \end{pmatrix}$ and $C(Y,Y,Y) = \begin{pmatrix} C_1(Y,Y,Y) \\ 0 \end{pmatrix}$
where

$$B_i(x,y) = \sum_{j,k=1}^{n} \frac{\partial^2 Y_i(\xi)}{\partial \xi_j \partial \xi_k}|_{\xi=0}(x_j y_k)$$

and

$$C_i(x,y,z) = \sum_{j,k,l=1}^{n} \frac{\partial^3 Y_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l}|_{\xi=0}(x_j y_k z_l)$$

$$B_1(\zeta,\eta) = -r(\zeta_1\eta_2) - r(\zeta_2\eta_1)$$

$$B_1(q,q) = -2(q_1q_2) - 2(q_2q_1) = -2(\frac{1+\sqrt{3}i}{2} + \frac{1+\sqrt{3}i}{2}) = -2(1+\sqrt{3}i)$$

$$B(q,q) = \begin{pmatrix} -2(1+\sqrt{3}i) \\ 0 \end{pmatrix}$$

$$B_1(q,\bar{q}) = -2(q_1\bar{q}_2 + q_2\bar{q}_1) = -2(\frac{1+\sqrt{3}i}{2} + \frac{1-\sqrt{3}i}{2}) = -2$$

$$B(q,\bar{q}) = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$B_1(\bar{q},\bar{q}) = -2(\bar{q}_1\bar{q}_2 + \bar{q}_2\bar{q}_1) = -2(\frac{1-\sqrt{3}i}{2} + \frac{1-\sqrt{3}i}{2}) = -2(1-\sqrt{3}i)$$

$$B(\bar{q},\bar{q}) = \begin{pmatrix} -2(1-\sqrt{3}i) \\ 0 \end{pmatrix}$$

$$C(\zeta,\eta,\xi) = 0$$

So

$$g_{20} =< p, B(q,q) >= \begin{pmatrix} \frac{-\sqrt{3}i}{3} \\ \frac{1}{2}+\frac{\sqrt{3}i}{6} \end{pmatrix} . \begin{pmatrix} -2(1+\sqrt{3}i) \\ 0 \end{pmatrix} = -2 + \frac{2\sqrt{3}i}{3}$$

$$g_{11} =< p, B(q,\bar{q}) >= \begin{pmatrix} \frac{-\sqrt{3}i}{3} \\ \frac{1}{2}+\frac{\sqrt{3}i}{6} \end{pmatrix} . \begin{pmatrix} -2 \\ 0 \end{pmatrix} = \frac{2\sqrt{3}i}{3}$$

$$g_{02} =< p, B(\bar{q},\bar{q}) >= \begin{pmatrix} \frac{-\sqrt{3}i}{3} \\ \frac{1}{2}+\frac{\sqrt{3}i}{6} \end{pmatrix} . \begin{pmatrix} -2(1-\sqrt{3}i) \\ 0 \end{pmatrix} = 2(1+\frac{\sqrt{3}i}{3})$$

$$g_{21} =< p, C(q,q,\bar{q} >= 0$$

The critical real part

$$a(0) = Re\left(\frac{e^{-i\theta_0}g_{21}}{2}\right) - Re\left(\frac{(1-2e^{i\theta_0})e^{-2i\theta_0}}{2(1-e^{i\theta_0})}g_{20}g_{11}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2$$

$$= 0 - Re\left(\frac{(1-2(\frac{1}{2}+\frac{\sqrt{3}i}{2}))(\frac{-1}{2}-\frac{\sqrt{3}i}{2})(-2+\frac{2\sqrt{3}i}{3})(\frac{2\sqrt{3}i}{3})}{2(1-(\frac{1}{2}+\frac{\sqrt{3}i}{2}))}\right) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2$$

$$= -Re\left(\frac{-4(-\sqrt{3}i)(\frac{-\sqrt{3}i}{2}-\frac{1}{2})(1-\frac{\sqrt{3}i}{3})(\sqrt{3}i)}{6(\frac{1}{2}-\frac{\sqrt{3}i}{2})}\right) - \frac{1}{2}(\frac{12}{9}) - \frac{1}{4}(4-\frac{12}{9})$$

$$= -Re\left(\frac{2(1+\sqrt{3}i)(1-\frac{\sqrt{3}i}{3})}{(1-\sqrt{3}i)}\right) - \frac{2}{3} - \frac{4}{3}$$

$$= -Re\left(\frac{2(1+\sqrt{3}i)(1-\frac{\sqrt{3}i}{3})(1+\sqrt{3}i)}{(1-\sqrt{3}i)(1+\sqrt{3}i)}\right) - 2$$

$$= -Re\left(\frac{4\sqrt{3}i}{3}\right) - 2 = -2$$

$$a(0) = -2 < 0$$

Therefore, a unique and stable closed invariant curve bifurcates from the nontrivial fixed point for $r > 2$.

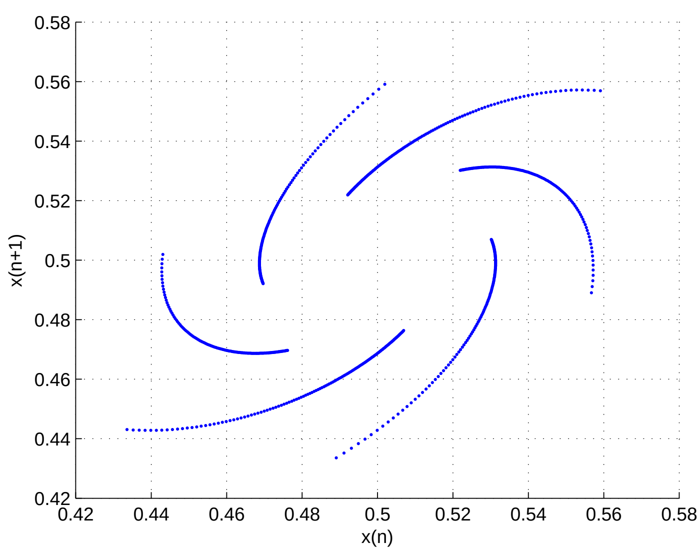

Fig. 3.2: Phase portrait at bifurcation value

In this thesis, we investigate the third order rational difference equation

$$X_{n+1} = \frac{\beta X_n + X_{n-2}}{A + X_{n-1}}$$

with positive parameters β, A and non-negative initial conditions X_0, X_{-1}, X_{-2}.
Also we investigate the fourth order rational equation

$$X_{n+1} = \frac{\beta X_n + X_{n-3}}{A + X_{n-1}}$$

with positive parameters β, A and non-negative initial conditions $X_0, X_{-1}, X_{-2}, X_{-3}$.

4. Third order rational difference equation

4.1 Introduction

Z.He and J.Qiu [9] studied the existence and direction of Neimark Sacker bifurcation of

$$X_{n+1} = \frac{\beta X_n + \alpha X_{n-2}}{1 + X_{n-1}} \tag{4.1.1}$$

and derived the following results:

Theorem 4.1.1. *Assume $\alpha > 1$ then the characteristic equation (4.1) has two complex roots that lie on the unit circle and another root lies inside it when $\beta = \beta^* = \frac{\alpha^2-\alpha}{\alpha+1}$, moreover the non-resonance and transversality conditions hold.*

Theorem 4.1.2. *Assume $\alpha > 1$ if $a(\beta^*) < 0$ (respectively > 0) then the Neimark Sacker bifurcation is supercritical (respectively subcritical) and unique closed invariant curve bifurcating from the positive fixed point is asymptotically stable (respectively unstable).*

4.2 Dynamics and Bifurcation of the third order equation

Consider the difference equation

$$X_{n+1} = \frac{\beta X_n + X_{n-2}}{A + X_{n-1}} \tag{4.2.1}$$

with positive parameters β, A and non-negative initial conditions X_0, X_{-1}, X_{-2}. We solve $f(x^*, x^*, x^*) = x^*$ to find its fixed points.

$$X^* = \frac{\beta X^* + X^*}{A + X^*}$$

$$X^*(A + X^*) = (\beta + 1)X^*$$
$$X^*(A + X^* - (\beta + 1)) = 0$$

So we have two fixed points,

$$X^* = (0, 0, 0), \quad X^* = (\beta - A + 1, \beta - A + 1, \beta - A + 1)$$

when $\beta + 1 > A$ we have a unique positive fixed point. Therefor, assume that $\beta + 1 > A$

Let $\begin{pmatrix} U_n \\ V_n \\ W_n \end{pmatrix} = \begin{pmatrix} X_n \\ X_{n-1} \\ X_{n-2} \end{pmatrix}$, then (4.2.1) is turned to

$$\begin{pmatrix} U_{n+1} \\ V_{n+1} \\ W_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\beta U_n + W_n}{A + V_n} \\ U_n \\ V_n \end{pmatrix} \tag{4.2.2}$$

Theorem 4.2.1. *The positive fixed point is stable if $\beta > \beta^*$, and unstable if $\beta < \beta^*$, where $\beta^* = \frac{1-A}{1+A}$*

Proof. The Jacobian matrix of (4.2.2) is

$$J = \begin{pmatrix} \frac{\partial U_{n+1}}{\partial U_n} & \frac{\partial U_{n+1}}{\partial V_n} & \frac{\partial U_{n+1}}{\partial W_n} \\ \frac{\partial V_{n+1}}{\partial U_n} & \frac{\partial V_{n+1}}{\partial V_n} & \frac{\partial V_{n+1}}{\partial W_n} \\ \frac{\partial W_{n+1}}{\partial U_n} & \frac{\partial W_{n+1}}{\partial V_n} & \frac{\partial W_{n+1}}{\partial W_n} \end{pmatrix}$$

Which equals

$$J = \begin{pmatrix} \frac{\beta}{A+V_n} & \frac{-(\beta U_n + W_n)}{(A+V_n)^2} & \frac{1}{A+V_n} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

At the positive fixed point

$$J = \begin{pmatrix} \frac{\beta}{A+(\beta-A+1)} & \frac{-(\beta+1)(\beta-A+1)}{(A+(\beta-A+1))^2} & \frac{1}{A+(\beta-A+1)} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J = \begin{pmatrix} \frac{\beta}{\beta+1} & \frac{-(\beta+1)(\beta-A+1)}{(\beta+1)^2} & \frac{1}{\beta+1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$J = \begin{pmatrix} \frac{\beta}{\beta+1} & \frac{-(\beta-A+1)}{\beta+1} & \frac{1}{\beta+1} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Let $P(\lambda)$ be the characteristic polynomial of the Jacobian matrix J.

$$P(\lambda) = (-1)^3|J - \lambda I|$$

$$P(\lambda) = \begin{vmatrix} \frac{\beta}{\beta+1} - \lambda & \frac{-(\beta-A+1)}{(\beta+1)} & \frac{1}{\beta+1} \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$P(\lambda) = \left[\left(\frac{\beta}{\beta+1} - \lambda\right)\lambda^2 - \left(\frac{-(\beta-A+1)}{\beta+1}\right)(-\lambda) + \frac{1}{\beta+1}\right] = 0$$

so

$$P(\lambda) = -\lambda^3 + \frac{\beta}{\beta+1}\lambda^2 - \frac{\beta-A+1}{\beta+1}\lambda + \frac{1}{\beta+1}. \tag{4.2.3}$$

Let $p(\lambda) = (-1)P(\lambda)$ then

$$p(\lambda) = \lambda^3 - \frac{\beta}{\beta+1}\lambda^2 + \frac{\beta - A + 1}{\beta+1}\lambda - \frac{1}{\beta+1}. \tag{4.2.4}$$

To study the stability of X^* we use Jury condition.
Jury's condition is an algebraic test, similar in form to the Routh - Hurwitz approach, that determines whether the roots of a polynomial lie within the unit circle. The test consists of two parts

1. simple test for necessary conditions
2. test for sufficient conditions

Theorem 4.2.2. *For a polynomial of the form:*

$$f(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

The necessary conditions for stability are:

$f(1) > 0$ *and* $(-1)^n f(-1) > 0$

The sufficient conditions for stability are obtained by forming a table as follows:

row	z^0	z^1	z^2	$\cdots$	z^{n-k}	$\cdots$	z^{n-1}	z^n
1	a_0	a_1	a_2	$\cdots$	a_{n-k}	$\cdots$	a_{n-1}	a_n
2	a_n	a_{n-1}	a_{n-2}	$\cdots$	a_k	$\cdots$	a_1	a_0
3	b_0	b_1	b_2	$\cdots$	b_{n-k}	$\cdots$	b_{n-1}	
4	b_{n-1}	b_{n-2}	b_{n-3}	$\cdots$	b_k	$\cdots$	b_0	
5	c_0	c_1	c_2	$\cdots$	$\cdots$	c_{n-2}		
6	c_{n-2}	c_{n-3}	c_{n-4}	$\cdots$	$\cdots$	c_0		
$\vdots$		$\vdots$		$\vdots$				
$2n-5$	P_0	P_1	P_2	P_3				
$2n-4$	P_3	P_2	P_1	P_0				
$2n-3$	q_0	q_1	q_2					

where $b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}$, $\quad c_k = \begin{vmatrix} b_0 & b_{n-1-k} \\ b_{n-1} & b_k \end{vmatrix}$

The sufficient conditions for stability are given by

$|a_0| < a_n, \quad |b_0| > |b_{n-1}|, \quad |c_0| > |c_{n-2}|, \cdots \ |q_0| > |q_2|$

Apply the necessary conditions,

$$p(1) = 1 - \frac{\beta}{\beta+1} + \frac{\beta - A + 1}{\beta+1} - \frac{1}{\beta+1}$$

$$p(1) = 1 - \frac{\beta+1}{\beta+1} + \frac{\beta - A + 1}{\beta+1} = \frac{\beta - A + 1}{\beta+1} > 0$$

$$(-1)^3 p(-1) = (-1)\left(-1 - \frac{\beta}{\beta+1} - \frac{\beta - A + 1}{\beta+1} - \frac{1}{\beta+1}\right)$$

$$(-1)^3 p(-1) = (-1)\left(-1 - \frac{\beta+1}{\beta+1} - \frac{\beta - A + 1}{\beta+1}\right)$$

$$(-1)^3 p(-1) = (-1)\left(-2 - \frac{\beta - A + 1}{\beta+1}\right) = 2 + \frac{\beta - A + 1}{\beta+1} > 0$$

The sufficient conditions are, $|a_0| < a_3$ and $|b_0| > |b_2|$

Where $a_0 = \frac{-1}{\beta+1}$, $\quad a_1 = \frac{\beta-A+1}{\beta+1}$, $\quad a_2 = \frac{-\beta}{\beta+1}$, $\quad a_3 = 1$

$b_0 = \begin{vmatrix} a_0 & a_3 \\ a_3 & a_0 \end{vmatrix}$ and $\quad b_2 = \begin{vmatrix} a_0 & a_1 \\ a_3 & a_2 \end{vmatrix}$

$|a_0| < a_3$ since

$$\left| -\frac{1}{\beta+1} \right| = \frac{1}{\beta+1} < 1$$

$$b_0 = \begin{vmatrix} \frac{-1}{\beta+1} & 1 \\ 1 & \frac{-1}{\beta+1} \end{vmatrix} = \frac{1}{(\beta+1)^2} - 1$$

but $\frac{1}{(\beta+1)^2} - 1 < 0$, so

$$|b_0| = 1 - \frac{1}{(\beta+1)^2} = \frac{(\beta+1)^2 - 1}{(\beta+1)^2} = \frac{\beta^2 + 2\beta}{(\beta+1)^2}$$

$$b_2 = \begin{vmatrix} \frac{-1}{\beta+1} & \frac{\beta-A+1}{\beta+1} \\ 1 & \frac{-\beta}{\beta+1} \end{vmatrix} = \frac{\beta}{(\beta+1)^2} - \frac{\beta - A + 1}{\beta+1}$$

$$b_2 = \frac{\beta - (\beta - A + 1)(\beta+1)}{(\beta+1)^2} = \frac{\beta - \beta^2 - 2\beta - 1 + A\beta + A}{(\beta+1)^2} = \frac{-\beta^2 - \beta - 1 + A\beta + A}{(\beta+1)^2}$$

We have two cases. If

$\frac{-\beta^2-\beta-1+A\beta+A}{(\beta+1)^2} > 0$, then $|b_2| = \frac{-\beta^2-\beta-1+A\beta+A}{(\beta+1)^2}$

We must have $|b_0| > |b_2|$ so

$$\frac{\beta^2+2\beta}{(\beta+1)^2} > \frac{-\beta^2-\beta-1+A\beta+A}{(\beta+1)^2}$$

$$\frac{2\beta^2+3\beta-A\beta-A+1}{(\beta+1)^2} > 0$$

$$\frac{(2\beta+1)(\beta+1)-A(\beta+1)}{(\beta+1)^2} > 0$$

$$\frac{(2\beta+1)-A}{(\beta+1)} > 0$$

The last inequality is satisfied since $\beta - A + 1 > 0$.
The second case is when

$$\frac{-\beta^2-\beta-1+A\beta+A}{(\beta+1)^2} < 0$$

So

$$|b_2| = \frac{\beta^2+\beta+1-A\beta-A}{(\beta+1)^2}$$

To have $|b_0| > |b_2|$

$$\frac{\beta^2+2\beta}{(\beta+1)^2} > \frac{\beta^2+\beta+1-A\beta-A}{(\beta+1)^2}$$

$$\frac{\beta+A\beta+A-1}{(\beta+1)^2} > 0$$

$$\frac{\beta(A+1)+A-1}{(\beta+1)^2} > 0$$

$$\beta > \frac{1-A}{1+A}$$

□

Theorem 4.2.3. *The difference equation* $X_{n+1} = \frac{\beta X_n + X_{n-2}}{A + X_{n-1}}$ *has no solution of period* 2.

Proof. By contradiction suppose it has period 2-solution say $\cdots p, q, p, q, \cdots$ where $p \neq q$. Then $p = \frac{\beta q + q}{A + p}$ so we have

$$q(\beta+1) = pA + p^2 \tag{4.2.5}$$

And $q = \frac{\beta p + p}{A + q}$ so we have

$$p(\beta+1) = qA + q^2 \tag{4.2.6}$$

Solving (4.2.5) and (4.2.6) we get

$$(p-q)(A+p+q+\beta+1) = 0$$

but$(A+p+q+\beta+1) > 0$ so $(p-q) = 0$ which implies $p = q$. A contradiction. □

4.3 Direction and stability of Neimark Sacker bifurcation

To determine the direction of the invariant closed curve that bifurcates from the positive fixed point we will follow the normal form theory of Neimark-Sacker bifurcation given in [1].

Theorem 4.3.1. *If $\beta = \beta^* = \frac{1-A}{1+A}$ then (4.2.4) has two complex conjugate roots that lie on the unit circle and another root lies inside the unit circle. Moreover for $A \in (0,1)$ the Neimark Sacker bifurcation conditions are satisfied.*

Proof. At first we will show that (4.2.4) has complex roots. We have $p(0) = \frac{-1}{\beta+1} < 0$ and $p(1) > 0$, then there exists $\zeta \in (0,1)$ such that $p(\zeta) = 0$. Moreover,

$$p'(\lambda) = 3\lambda^2 - \frac{2\beta}{\beta+1}\lambda + \frac{\beta - A + 1}{\beta + 1}$$

The discriminant of $p'(\lambda)$ is

$$\begin{aligned}\Delta p'(\lambda) &= \left(\frac{2\beta}{\beta+1}\right)^2 - 4(3)\left(\frac{\beta - A + 1}{\beta+1}\right)\\ &= \frac{4\beta^2 - 12(\beta - A + 1)(\beta+1)}{(\beta+1)^2}\end{aligned}$$

$$= \frac{4\beta^2 - 12(\beta+1)^2 + 12A(\beta+1)}{(\beta+1)^2}$$

$$= \frac{4\beta^2 - 12\beta^2 - 24\beta + 12A\beta + 12A - 12}{(\beta+1)^2}$$

$$= \frac{-8\beta^2 + 12A\beta - 24\beta + 12A - 12}{(\beta+1)^2}$$

$$= \frac{-8\beta^2 + 12(\beta A + A - 1) - 24\beta}{(\beta+1)^2}$$

Using $\beta(A+1) + A - 1 = 0$.

$$\Delta p'(\lambda) = \frac{-8\beta^2 + 12(-\beta) - 24\beta}{(\beta+1)^2}$$

$$\Delta p'(\lambda) = \frac{-8\beta^2 - 36\beta}{(\beta+1)^2} < 0$$

So $p(\lambda)$ doesn't change its direction, hence there exists two conjugate complex roots of $p(\lambda)$

Next we show that (4.2.4) has two conjugate complex roots on the unit circle when $\beta = \beta^*$, using the next theorem.

Theorem 4.3.2. *(Viète formula) [1]*
For any general polynomial of degree n

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

Viète formulas relate the polynomial's coefficients a_k to signed sums and products of its roots z_i , $i = 1, 2, \cdots, n$ as follows

$$z_1 + z_2 + \cdots + z_{n-1} + z_n = \frac{-a_{n-1}}{a_n}$$

$$(z_1 z_2 + z_1 z_3 + \cdots + z_1 z_n) + (z_2 z_3 + z_2 z_4 + \cdots + z_2 z_n) + \cdots + z_{n-1} z_n = \frac{a_{n-2}}{a_n}$$

$$\vdots$$

$$z_1 z_2 \cdots z_n = (-1)^n \frac{a_0}{a_n}$$

Suppose that $\lambda_1, \lambda_2, \lambda_3$ are three roots of (4.2.4) where $\lambda_1 = \bar{\lambda_2}$ and $\lambda_3 = \zeta$. By *Viète* theorem to the polynomial

$$p(\lambda) = \lambda^3 - \frac{\beta}{\beta+1}\lambda^2 + \frac{\beta - A + 1}{\beta+1}\lambda - \frac{1}{\beta+1}$$

where $a_0 = \frac{-1}{\beta+1}$, $a_1 = \frac{\beta-A+1}{\beta+1}$, $a_2 = \frac{-\beta}{\beta+1}$ $a_3 = 1$. If $|\lambda_1| = |\lambda_2| = 1$ and $\lambda_3 = \zeta$ we obtain

$$\lambda_1 + \lambda_2 + \lambda_3 = \frac{-a_2}{a_3} = \frac{\beta}{\beta+1} \tag{4.3.1}$$

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = \frac{a_1}{a_3} = \frac{\beta - A + 1}{\beta+1} \tag{4.3.2}$$

$$\lambda_1\lambda_2\lambda_3 = \frac{-a_0}{a_3} = \frac{1}{\beta+1} \tag{4.3.3}$$

from (4.3.3)

$$\lambda_1\lambda_2\lambda_3 = \lambda_3 = \frac{1}{\beta+1}$$

substitute λ_3 in (4.3.2) and note that $\lambda_1\lambda_2 = 1$

$$1 + (\lambda_1 + \lambda_2)\frac{1}{\beta+1} = \frac{\beta - A + 1}{\beta+1}.$$

$$(\beta+1) + (\lambda_1 + \lambda_2) = \beta - A + 1$$

So

$$\lambda_1 + \lambda_2 = -A$$

substitute λ_3 in (4.3.1)

$$\lambda_1 + \lambda_2 = \frac{\beta}{\beta+1} - \frac{1}{\beta+1} = \frac{\beta-1}{\beta+1}.$$

So

$$\lambda_1 + \lambda_2 = \frac{\beta-1}{\beta+1} = -A$$

which implies that

$$\beta = \frac{1-A}{1+A} = \beta^*$$

Since the roots are uniquely determined, the above argument implies the existence of conjugate pair of complex roots on the unit circle.

Let $e^{i\theta}, e^{-i\theta}$ be the roots of $p(\lambda)$ at β^* then,

$$e^{3i\theta} - \frac{\beta}{\beta+1}e^{2i\theta} + \frac{\beta - A + 1}{\beta+1}e^{i\theta} - \frac{1}{\beta+1} = 0.$$

$$\cos 3\theta + i\sin 3\theta - \frac{\beta}{\beta+1}(\cos 2\theta + i\sin 2\theta) + \frac{\beta - A + 1}{\beta+1}(\cos\theta + i\sin\theta) - \frac{1}{\beta+1} = 0.$$

Separate the real part and imaginary parts;

$$\cos 3\theta - \frac{\beta}{\beta+1}\cos 2\theta + \frac{\beta - A + 1}{\beta+1}\cos\theta - \frac{1}{\beta+1} = 0.$$

$$\sin 3\theta - \frac{\beta}{\beta+1}\sin 2\theta + \frac{\beta - A + 1}{\beta+1}\sin\theta = 0.$$

rewrite the two equations in the form,

$$\cos 3\theta - \frac{\beta}{\beta+1}\cos 2\theta = -\frac{\beta - A + 1}{\beta+1}\cos\theta + \frac{1}{\beta+1}.$$

$$\sin 3\theta - \frac{\beta}{\beta+1}\sin 2\theta = -\frac{\beta - A + 1}{\beta+1}\sin\theta.$$

Square both sides of equations,
$\cos^2 3\theta + \left(\frac{\beta}{\beta+1}\right)^2\cos^2 2\theta - \left(\frac{2\beta}{\beta+1}\right)\cos 2\theta\cos 3\theta = \left(\frac{1}{\beta+1}\right)^2 + \left(\frac{\beta-A+1}{\beta+1}\right)^2\cos^2\theta - \frac{2(\beta-A+1)}{(\beta+1)^2}\cos\theta.$

$$\sin^2 3\theta + \left(\frac{\beta}{\beta+1}\right)^2\sin^2 2\theta - \frac{2\beta}{\beta+1}\sin 3\theta\sin 2\theta = \left(\frac{\beta - A + 1}{\beta+1}\right)^2\sin^2\theta.$$

Then add them to each other,
$\cos^2 3\theta + \left(\frac{\beta}{\beta+1}\right)^2\cos^2 2\theta - \frac{2\beta}{\beta+1}\cos 2\theta\cos 3\theta + \sin^2 3\theta + \left(\frac{\beta}{\beta+1}\right)^2\sin^2\theta - \frac{2\beta}{\beta+1}\sin 3\theta\sin 2\theta =$

$\left(\frac{1}{\beta+1}\right)^2+\left(\frac{\beta-A+1}{\beta+1}\right)^2\cos^2\theta-\frac{2(\beta-A+1)}{(\beta+1)^2}\cos\theta+\left(\frac{\beta-A+1}{\beta+1}\right)^2\sin^2\theta$

$\cos^2 3\theta+\sin^2 3\theta+\left(\frac{\beta}{\beta+1}\right)^2(\cos^2 2\theta+\sin^2 2\theta)-\frac{2\beta}{\beta+1}(\cos 2\theta\cos 3\theta+\sin 2\theta\sin 3\theta)=$
$\left(\frac{1}{\beta+1}\right)^2+\left(\frac{\beta-A+1}{\beta+1}\right)^2(\cos^2\theta+\sin^2\theta)-\frac{2(\beta-A+1)}{(\beta+1)^2}\cos\theta$

$$1+\left(\frac{\beta}{\beta+1}\right)^2-\left(\frac{1}{\beta+1}\right)^2-\left(\frac{\beta-A+1}{\beta+1}\right)^2=\left(\frac{2\beta}{\beta+1}-\frac{2(\beta-A+1)}{(\beta+1)^2}\right)\cos\theta.$$

Thus,

$$(\beta+1)^2+\beta^2-1-(\beta-A+1)^2=(2\beta(\beta+1)-2(\beta-A+1))\cos\theta$$

$$(\beta+1)^2+\beta^2-1-(\beta+1)^2+2A(\beta+1)-A^2=(2\beta^2+2\beta-2\beta-2+2A)\cos\theta$$
$$\beta^2-A^2-1+2A(\beta+1)=(2\beta^2-2+2A)\cos\theta$$

At $\beta^*=\beta=\frac{1-A}{1+A}$, $\quad 2A(\beta+1)=2-2\beta$, we have

$$\beta^2-A^2-1+2-2\beta=(2\beta^2-2+2A)\cos\theta$$

$$\beta^2-A^2-2\beta+1=(2\beta^2-2+2A)\cos\theta$$
$$\beta^2-2\beta+1-A^2=(2\beta^2-2+2A)\cos\theta$$
$$\cos\theta=\frac{(\beta-1)^2-A^2}{2\beta^2+2A-2}=\frac{(\beta-A-1)(\beta+A-1)}{2\beta^2-2A\beta-2\beta}\;|_{\beta=\frac{1-A}{1+A}}$$
$$=\frac{(\beta-A-1)(\beta+A-1)}{2\beta(\beta-A-1)}=\frac{\beta+A-1}{2\beta}=\frac{-A}{2}$$

then for $A\in(0,1)$, $\quad \frac{-1}{2}<\cos\theta<0$. So there exists $\theta^*\in(\frac{\pi}{2},\pi)$ such that

$$\theta_0=\cos^{-1}\left(\frac{-A}{2}\right)$$

Note that $e^{ik\theta^*}\neq 1$ for $A\in(0,1)$ where $k=1,2,3,4$
Next we will show that $\frac{d|\lambda|^2}{d\beta}\;|_{\beta=\beta^*}\neq 0$.

$$p(\lambda)=\lambda^3-\frac{\beta}{\beta+1}\lambda^2+\frac{\beta-A+1}{\beta+1}\lambda-\frac{1}{\beta+1}$$

$$\frac{d|\lambda|^2}{d\beta}\;|_{\beta=\beta^*}=\frac{d(\lambda\bar{\lambda})}{d\beta}=\lambda\frac{\partial\bar{\lambda}}{\partial\beta}+\bar{\lambda}\frac{\partial\lambda}{\partial\beta}$$

$$\frac{d|\lambda|^2}{d\beta}\mid_{\beta=\beta^*}=\lambda\left(\frac{\partial p(\bar{\lambda})}{\partial\beta}.\frac{\partial\bar{\lambda}}{\partial p(\bar{\lambda})}\right)+\bar{\lambda}\left(\frac{\partial p(\lambda)}{\partial\beta}.\frac{\partial\lambda}{\partial p(\lambda)}\right)$$

$$=\lambda\left(\frac{\frac{-(\beta+1)+\beta}{(\beta+1)^2}\bar{\lambda}^2+\frac{(\beta+1)-(\beta-A+1)}{(\beta+1)^2}\bar{\lambda}+\frac{1}{(\beta+1)^2}}{3\bar{\lambda}^2-\frac{2\beta}{\beta+1}\bar{\lambda}+\frac{\beta-A+1}{\beta+1}}\right)+\bar{\lambda}\left(\frac{\frac{-(\beta+1)+\beta}{(\beta+1)^2}\lambda^2+\frac{(\beta+1)-(\beta-A+1)}{(\beta+1)^2}\lambda+\frac{1}{(\beta+1)^2}}{3\lambda^2-\frac{2\beta}{\beta+1}\lambda+\frac{\beta-A+1}{\beta+1}}\right)$$

$$=\lambda\left(\frac{-1\bar{\lambda}^2+A\bar{\lambda}+1}{(\beta+1)^2(3\bar{\lambda}^2-\frac{2\beta}{\beta+1}\bar{\lambda}+\frac{\beta-A+1}{\beta+1}}\right)+\bar{\lambda}\left(\frac{-1\lambda^2+A\lambda+1}{(\beta+1)^2(3\lambda^2-\frac{2\beta}{\beta+1}\lambda+\frac{\beta-A+1}{\beta+1}}\right)$$

$$=\left(\frac{(-1)\bar{\lambda}+A+\lambda}{(\beta+1)(3(\beta+1)\bar{\lambda}^2-2\beta\bar{\lambda}+\beta-A+1)}\right)+\left(\frac{(-1)\lambda+A+\bar{\lambda}}{(\beta+1)(3(\beta+1)\lambda^2-2\beta\lambda+\beta-A+1)}\right)$$

$$=\left(\frac{A+2i\sin\theta}{(\beta+1)(3(\beta+1)\bar{\lambda}^2-2\beta\bar{\lambda}+\beta-A+1)}\right)+\left(\frac{A-2i\sin\theta}{(\beta+1)(3(\beta+1)\lambda^2-2\beta\lambda+\beta-A+1)}\right)$$

$$=\frac{(A+2i\sin\theta)(3(\beta+1)\lambda^2-2\beta\lambda+\beta-A+1)+(A-2i\sin\theta)(3(\beta+1)\bar{\lambda}^2-2\beta\bar{\lambda}+\beta-A+1)}{(\beta+1)(3(\beta+1)\bar{\lambda}^2-2\beta\bar{\lambda}+\beta-A+1)(3(\beta+1)\lambda^2-2\beta\lambda+\beta-A+1)}$$

$$=\frac{3A(\beta+1)(\bar{\lambda}^2+\lambda^2)-2\beta A(\lambda+\bar{\lambda})+6i(\beta+1)\sin\theta(\lambda^2-\bar{\lambda}^2)+4i\beta\sin\theta(\bar{\lambda}-\lambda)+2A(\beta-A+1)}{(\beta+1)(3(\beta+1)\bar{\lambda}^2-2\beta\bar{\lambda}+\beta-A+1)(3(\beta+1)\lambda^2-2\beta\lambda+\beta-A+1)}$$

But

$$\lambda+\bar{\lambda}=(\cos\theta+i\sin\theta)+(\cos\theta-i\sin\theta)=2\cos\theta$$

$$\lambda^2+\bar{\lambda}^2=(\cos\theta+i\sin\theta)^2+(\cos\theta-i\sin\theta)^2=2\cos^2\theta-2\sin^2\theta$$

$$\lambda^2-\bar{\lambda}^2=(\cos\theta+i\sin\theta)^2-(\cos\theta-i\sin\theta)^2=4i\cos\theta\sin^2\theta$$

So $\frac{d|\lambda|^2}{d\beta}\mid_{\beta=\beta^*}=$

$$\frac{6A(\beta+1)(\cos^2\theta-\sin^2\theta)-4\beta A\cos\theta+6i(\beta+1)\sin\theta(4i\cos\theta\sin\theta)+8\beta\sin^2\theta+2A(\beta-A+1)}{(\beta+1)(3(\beta+1)\bar{\lambda}^2-2\beta\bar{\lambda}+\beta-A+1)(3(\beta+1)\lambda^2-2\beta\lambda+\beta-A+1)}$$

$$=\frac{6A(\beta+1)(2\cos^2\theta-1)-4\beta A\cos\theta+(-24(\beta+1)\cos\theta+8\beta)\sin^2\theta+2A(\beta-A+1)}{(\beta+1)(3(\beta+1)\bar{\lambda}^2-2\beta\bar{\lambda}+\beta-A+1)(3(\beta+1)\lambda^2-2\beta\lambda+\beta-A+1)}$$

$$= \frac{6A(\beta+1)(2\cos^2\theta-1)-4\beta A\cos\theta+(-24(\beta+1)\cos\theta+8\beta)(1-\cos^2\theta)+2A(\beta-A+1)}{(\beta+1)(3(\beta+1)\bar{\lambda}^2-2\beta\bar{\lambda}+\beta-A+1)(3(\beta+1)\lambda^2-2\beta\lambda+\beta-A+1)}$$

At $\beta^* = \beta = \frac{1-A}{1+A}$, $\cos\theta = \cos\theta_0 = \frac{-A}{2}$, we get

$$\frac{d|\lambda|^2}{d\beta}\,|_{\beta=\beta^*}= \frac{6A(\beta+1)(\frac{2A^2}{4}-1)+4\beta A\frac{A}{2}+(24(\beta+1)\frac{A}{2}+8\beta)(1-\frac{A^2}{4})+2A(\beta-A+1)}{(\beta+1)(3(\beta+1)\bar{\lambda}^2-2\beta\bar{\lambda}+\beta-A+1)(3(\beta+1)\lambda^2-2\beta\lambda+\beta-A+1)}$$

$$= \frac{6A(\beta+1)(\frac{A^2}{2}-1)+2\beta A^2+2A(\beta-A+1)+(12(\beta+1)A+8\beta)(1-\frac{A^2}{4})}{(\beta+1)(3(\beta+1)\bar{\lambda}^2-2\beta\bar{\lambda}+\beta-A+1)(3(\beta+1)\lambda^2-2\beta\lambda+\beta-A+1)}$$

$$= \frac{3A^3(\beta+1)-6A(\beta+1)+2\beta A^2+2A(\beta+1)-2A^2+12A(\beta+1)+8\beta-3A^3(\beta+1)-2\beta A^2}{(\beta+1)(3(\beta+1)\bar{\lambda}^2-2\beta\bar{\lambda}+\beta-A+1)(3(\beta+1)\lambda^2-2\beta\lambda+\beta-A+1)}$$

$$= \frac{8A(\beta+1)-2A^2+8\beta}{(\beta+1)(3(\beta+1)\bar{\lambda}^2-2\beta\bar{\lambda}+\beta-A+1)(3(\beta+1)\lambda^2-2\beta\lambda+\beta-A+1)}$$

Substitute $\beta = \frac{1-A}{1+A}$

$$\frac{d|\lambda|^2}{d\beta}\,|_{\beta=\beta^*}= \frac{8A\frac{2}{1+A}-2A^2+8\frac{(1-A)}{1+A}}{(\beta+1)(3(\beta+1)\bar{\lambda}^2-2\beta\bar{\lambda}+\beta-A+1)(3(\beta+1)\lambda^2-2\beta\lambda+\beta-A+1)}$$

$$= \frac{16A-2A^2(1+A)+8(1-A)}{(1+A)(\beta+1)(3(\beta+1)\bar{\lambda}^2-2\beta\bar{\lambda}+\beta-A+1)(3(\beta+1)\lambda^2-2\beta\lambda+\beta-A+1)}$$

$$= \frac{-2A^3-2A^2+8A+8}{(1+A)(\beta+1)(3(\beta+1)\bar{\lambda}^2-2\beta\bar{\lambda}+\beta-A+1)(3(\beta+1)\lambda^2-2\beta\lambda+\beta-A+1)}$$

$$= \frac{(A-2)(A+2)(A+1)}{(1+A)(\beta+1)(3(\beta+1)\bar{\lambda}^2-2\beta\bar{\lambda}+\beta-A+1)(3(\beta+1)\lambda^2-2\beta\lambda+\beta-A+1)}$$

$\frac{d|\lambda|^2}{d\beta}\,|_{\beta=\beta^*}= 0$ if $A=-1,-2$ but this contradicts the assumption that A is positive parameter.

And $\frac{d|\lambda|^2}{d\beta}\,|_{\beta=\beta^*}= 0$ if $A = 2$ but $2 \notin (0,1)$. So $\frac{d|\lambda|^2}{d\beta}\,|_{\beta=\beta^*}\neq 0$ in the interval $(0,1)$ □

Now we shift the fixed point to the origin by taking

$\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} = \begin{pmatrix} U_n - U^* \\ V_n - V^* \\ W_n - W^* \end{pmatrix}$, equation (4.2.2) becomes,

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\beta(x_n+X^*)+z_n+X^*}{A+y_n+X^*} - X^* \\ x_n \\ y_n \end{pmatrix} \tag{4.3.4}$$

Equation (4.3.4) can be written as

$$Y_{n+1} = JY_n + G(Y_n) \tag{4.3.5}$$

where $G(Y) = \frac{1}{2}B(Y,Y) + \frac{1}{6}C(Y,Y,Y) + O(||Y||^3)$ and $Y_n = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$

$$B(Y,Y) = \begin{pmatrix} B_1(Y,Y) \\ 0 \\ 0 \end{pmatrix} \quad \text{and } C(Y,Y,Y) = \begin{pmatrix} C_1(Y,Y,Y) \\ 0 \\ 0 \end{pmatrix}$$

where

$$B_i(x,y) = \sum_{j,k=1}^{n} \frac{\partial^2 Y_i(\xi)}{\partial \xi_j \partial \xi_k}|_{\xi=0}(x_j y_k)$$

and

$$C_i(x,y,z) = \sum_{j,k,l=1}^{n} \frac{\partial^3 Y_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l}|_{\xi=0}(x_j y_k z_l)$$

$$B_1(\phi,\psi) = 2\frac{(\beta - A + 1)}{(\beta+1)^2}\phi_2\psi_2 - \frac{\beta}{(\beta+1)^2}(\phi_1\psi_2 + \phi_2\psi_1) - \frac{1}{(\beta+1)^2}(\phi_2\psi_3 + \phi_3\psi_2)$$

$$\begin{aligned} C_1(\phi,\psi,\eta) = -6\frac{(\beta - A + 1)}{(\beta+1)^3}\phi_2\psi_2\eta_2 + \frac{2\beta}{(\beta+1)^3}(\phi_2\psi_2\eta_3 + \phi_3\psi_2\eta_2 + \phi_2\psi_3\eta_2) \\ + \frac{2}{(\beta+1)^3}(\phi_2\psi_2\eta_1 + \phi_1\psi_2\eta_2 + \phi_2\psi_1\eta_2) \end{aligned}$$

Now, we find the eigenvectors of J and J^* corresponding to $e^{\pm i\theta_0}$ at $\theta_0 =$

$\theta = \cos^{-1}(\frac{-A}{2})$.
Let $Jq = e^{i\theta_0}q$, $J^T p^* = e^{-i\theta_0}p^*$ where q and p^* are the eigenvectors corresponding to the eigenvalues $e^{i\theta_0}$ and $e^{-i\theta_0}$, respectively .

Solving $(J-\lambda I)q = (J - e^{i\theta_0}I)q = 0$

$$\begin{pmatrix} \frac{\beta}{\beta+1} - e^{i\theta_0} & -\frac{(\beta-A+1)}{\beta+1} & \frac{1}{\beta+1} \\ 1 & -e^{i\theta_0} & 0 \\ 0 & 1 & -e^{i\theta_0} \end{pmatrix} q = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Let $q_1 = 1$, from the second equation

$$1q_1 + (-e^{i\theta_0})q_2 = 0$$

So $q_2 = e^{-i\theta_0}$, and from the third equation we get

$$q_2 + (-e^{i\theta_0})q_3 = 0$$

So $q_3 = e^{-2i\theta_0}$
Thus we obtain $q \sim \begin{pmatrix} 1 \\ e^{-i\theta_0} \\ e^{-2i\theta_0} \end{pmatrix}$
Note that this choice of q satisfies the first equation too. To have a non zero solution of the system $(J-\lambda I)q = 0$, the matrix $(J-\lambda I)$ must be singular, that means $|J-\lambda I| = 0$

$$|J-\lambda I| = \left(\frac{\beta}{\beta+1} - e^{i\theta_0}\right)e^{2i\theta_0} + \frac{\beta-A+1}{\beta+1}e^{i\theta} + \frac{1}{\beta+1}$$

$$= e^{2i\theta_0}\left(\frac{\beta}{\beta+1} - e^{i\theta_0} - \frac{\beta-A+1}{\beta+1}e^{-i\theta_0} + \frac{1}{\beta+1}e^{-2i\theta_0}\right) = 0$$

So we have

$$\frac{\beta}{\beta+1} - e^{i\theta_0} + \frac{\beta-A+1}{\beta+1}e^{-i\theta_0} + \frac{1}{\beta+1}e^{-2i\theta_0} = 0$$

Also, solving $(J-\lambda I)^T p^* = (J - e^{-i\theta_0}I)^T p^* = 0$

$$\begin{pmatrix} \frac{\beta}{\beta+1} - e^{-i\theta_0} & 1 & 0 \\ -\frac{(\beta-A+1)}{\beta+1} & -e^{-i\theta_0} & 1 \\ \frac{1}{\beta+1} & 0 & -e^{-i\theta_0} \end{pmatrix} p^* = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Let $p_1^* = 1$, from the first equation

$$\frac{\beta}{\beta+1} - e^{-i\theta_0} + p_2^* = 0$$

So $p_2^* = e^{-i\theta_0} - \frac{\beta}{\beta+1}$, from the third equation we get

$$\frac{1}{\beta+1} - e^{-i\theta_0} p_3^* = 0$$

So $p_3^* = \frac{e^{i\theta_0}}{\beta+1}$

Thus we obtain $p^* \sim \begin{pmatrix} 1 \\ e^{-i\theta_0} - \frac{\beta}{\beta+1} \\ \frac{e^{i\theta_0}}{\beta+1} \end{pmatrix}$
Note that this choice of p^* also satisfies the second equation.

To normalize p^* and q, we must have $< p^*, q >= 1$, where $< .,. >$ is the standard scalar product in $\mathbb{C}^3$.

$$< p^*, q >= \sum_{i=1}^{3} \overline{p_i^*} q_i = \begin{pmatrix} 1 \\ e^{i\theta_0} - \frac{\beta}{\beta+1} \\ \frac{e^{-i\theta_0}}{\beta+1} \end{pmatrix} \begin{pmatrix} 1 \\ e^{-i\theta_0} \\ e^{-2i\theta_0} \end{pmatrix}$$

$$= 1 + e^{-i\theta_0}\left(e^{i\theta_0} - \frac{\beta}{\beta+1}\right) + e^{-2i\theta_0}\frac{e^{-i\theta_0}}{\beta+1} = 2 - \frac{\beta e^{-i\theta_0}}{\beta+1} + \frac{e^{-3i\theta_0}}{\beta+1}$$

So let $p = \eta * \begin{pmatrix} 1 \\ e^{-i\theta} - \frac{\beta}{\beta+1} \\ \frac{e^{i\theta}}{\beta+1} \end{pmatrix}$ where $\eta = (2 - \frac{\beta e^{-i\theta_0}}{\beta+1} + \frac{e^{-3i\theta_0}}{\beta+1})^{-1}$

The critical real eigenspace T^c corresponding to $\lambda_{1,2}$ is two-dimensional and is spanned by $\{\mathbb{R}e(q), \mathbb{I}m(q)\}$. The real eigenspace T^s corresponding to the real eigenvalues of J is one-dimensional. Any vector $x \in \mathbb{R}^3$ may be decomposed as

$$x = zq + \bar{z}\bar{q} + y$$

where $z \in \mathbb{C}^1$, and $\bar{z}\bar{q} \in T^c$, $y \in T^s$. The complex variable z is a coordinate on T^c. We have

$$\begin{cases} z =< p, x > \\ y = x- < p, x > q- < \bar{p}, x > \bar{q} \end{cases}$$

In these coordinates, the map (5.3.14) takes the form

$$\begin{cases} \tilde{z} = e^{i\theta_0}z + < p, G(zq + \bar{z}\bar{q} + y) > \\ \tilde{y} = Jy + G(zq + \bar{z}\bar{q} + y) - < p, G(zq + \bar{z}\bar{q} + y) > q - < \bar{p}, G(zq + \bar{z}\bar{q} + y) > \bar{q} \end{cases}$$

The previous system can be written in the form:

$$\begin{cases} \tilde{z} = e^{i\theta_0}z + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2 + \frac{1}{2}G_{21}z^2\bar{z} + < G_{10}, y > z + < G_{01}, y > \bar{z} \\ \tilde{y} = Jy + \frac{1}{2}H_{20}z^2 + H_{11}z\bar{z} + \frac{1}{2}H_{02}\bar{z}^2 + \frac{1}{2}H_{21}z^2\bar{z} \end{cases}$$

Where $\begin{cases} G_{20} =< p, B(q,q) >, G_{11} =< p, B(q,\bar{q}) >, \\ G_{02} =< p, B(\bar{q},\bar{q}) >, G_{21} =< p, C(q,q,\bar{q}) > \end{cases}$

$$\begin{cases} H_{20} = B(q,q) - < p, B(q,q) > q - < \bar{p}, B(q,q) > \bar{q} \\ H_{11} = B(q,\bar{q}) - < p, B(q,\bar{q}) > q - < \bar{p}, B(q,\bar{q}) > \bar{q} \end{cases}$$

$\Big\{ < G_{10}, y >=< p, B(q,y) >, < G_{01}, y >=< p, B(\bar{q},y) >$

And the scalar product in $\mathbb{C}^3$ is used.

From the center manifold theorem, there exists a center manifold W^c which can be approximated as

$$Y = V(z,\bar{z}) = \frac{1}{2}w_{20}z^2 + w_{11}z\bar{z} + \frac{1}{2}w_{02}\bar{z}^2$$

where $< q, w_{ij} >= 0$. The vectors $w_{ij} \in \mathbb{C}^3$ can be found from the linear equations

$$\begin{cases} w_{20} = (e^{2i\theta_0}I_3 - J)^{-1}H_{20} \\ w_{11} = (I_3 - J)^{-1}H_{11} \\ w_{02} = (e^{-2i\theta_0}I_3 - J)^{-1}H_{02} \end{cases}$$

So z can be expressed as

$$\tilde{z} = e^{i\theta_0}\bar{z} + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2$$

$$+\frac{1}{2}(G_{21} + 2 < p, B(q, (I-J)^{-1}H_{11}) > + < p, B(\bar{q}, (e^{2i\theta_0}I - J)^{-1}H_{20}) >)z^2\bar{z}$$

Taking into account the identities

$$(I-J)^{-1}q = \frac{1}{1-e^{i\theta_0}}q, \qquad (e^{2i\theta_0}I-J)^{-1}q = \frac{e^{-i\theta_0}}{e^{i\theta_0}-1}q$$

and

$$(I-J)^{-1}\bar{q} = \frac{1}{1-e^{i\theta_0}}\bar{q}, \qquad (e^{2i\theta_0}I-J)^{-1}\bar{q} = \frac{e^{-i\theta_0}}{e^{i\theta_0}-1}\bar{q}$$

also we can express z using the map

$$\widetilde{z} = e^{i\theta_0}z + \sum_{k+l\geq 2}\frac{1}{k!j!}g_{k_j}z^k\bar{z}^j$$

where

$$g_{20} =< p, B(q,q) >, \qquad g_{11} =< p, B(q,\bar{q}) >, \qquad g_{02} =< p, B(\bar{q},\bar{q}) >$$

$$g_{21} =< p, C(q,q,\bar{q}) > +2 < p, B(q,(I-J)^{-1}B(q,\bar{q})) > +$$

$$< p, B(\bar{q},(e^{2i\theta_0}I-J)^{-1}B(q,q)) > +\frac{e^{-i\theta_0}(1-2e^{i\theta_0})}{1-e^{i\theta_0}} < p, B(q,q) >< p, B(q,\bar{q}) >$$

$$-\frac{2}{1-e^{-i\theta_0}}| < p, B(q,\bar{q}) > |^2 - \frac{e^{i\theta_0}}{e^{3i\theta_0}-1}| < p, B(\bar{q},\bar{q}) > |^2$$

Or equivalently

$$\widetilde{z} = e^{i\theta_0}z(1+d(\beta^*))|z^2|$$

where the real number $A(\beta^*) = \mathbb{R}e(d(\beta^*))$ that determines the direction of bifurcation of a closed invariant curve, can be computed via

$$A(\beta^*) = \mathbb{R}e(\frac{e^{-i\theta_0}g_{21}}{2}) - \mathbb{R}e(\frac{(1-2e^{i\theta_0})e^{-2i\theta_0}}{2(1-e^{i\theta_0})}g_{20}g_{11}) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2$$

Where $g_{20} =< p, B(q,q) >$

$$B(q,q) = \begin{pmatrix} \frac{2(\beta-A+1)e^{-2i\theta_0}-2\beta e^{-i\theta_0}-2e^{-3i\theta_0}}{(\beta+1)^2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} L \\ 0 \\ 0 \end{pmatrix}$$

$$g_{20} = \frac{\beta+1}{2(\beta+1)-\beta e^{i\theta_0}+e^{3i\theta_0}} \frac{2(\beta-A+1)e^{-2i\theta_0}-2\beta e^{-i\theta_0}-2e^{-3i\theta_0}}{(\beta+1)^2}$$

$$g_{20} = \frac{2(\beta-A+1)e^{-2i\theta_0}-2\beta e^{-i\theta_0}-2e^{-3i\theta_0}}{(\beta+1)\left(2(\beta+1)-\beta e^{i\theta_0}+e^{3i\theta_0}\right)}$$

$$g_{11} =< p, B(q,\bar{q}) >, \qquad B_1(q,\bar{q}) = \frac{2(\beta-A+1)-\beta(e^{i\theta_0}+e^{-i\theta_0})-(e^{i\theta_0}+e^{-i\theta_0})}{(\beta+1)^2}$$

$$B(q,\bar{q}) = \begin{pmatrix} \frac{2(\beta-A+1)-2(\beta+1)\cos\theta_0}{(\beta+1)^2} \\ 0 \\ 0 \end{pmatrix}$$

$$g_{11} = \frac{2(\beta-A+1)-2(\beta+1)\cos\theta_0}{(\beta+1)^2} * \frac{\beta+1}{2(\beta+1)-\beta e^{i\theta_0}+e^{3i\theta_0}}$$

$$g_{11} = \frac{2(\beta-A+1)-2(\beta+1)\cos\theta_0}{(\beta+1)\left(2(\beta+1)-\beta e^{i\theta_0}+e^{3i\theta_0}\right)}$$

$$g_{02} =< p, B(\bar{q},\bar{q}) >, \qquad B(\bar{q},\bar{q}) = \begin{pmatrix} \frac{2(\beta-A+1)e^{2i\theta_0}-2\beta e^{i\theta_0}-2e^{3i\theta_0}}{(\beta+1)^2} \\ 0 \\ 0 \end{pmatrix}$$

$$g_{02} = \frac{2(\beta-A+1)e^{2i\theta_0}-2\beta e^{i\theta_0}-2e^{3i\theta_0}}{(\beta+1)^2} \frac{\beta+1}{2(\beta+1)-\beta e^{i\theta_0}+e^{3i\theta_0}}$$

$$g_{02} = \frac{2(\beta-A+1)e^{2i\theta_0}-2\beta e^{i\theta_0}-2e^{3i\theta_0}}{(\beta+1)\left(2(\beta+1)-\beta e^{i\theta_0}+e^{3i\theta_0}\right)}$$

$$g_{21} =< p, C(q,q,\bar{q}) > +2 < p, B(q,(I-J)^{-1}B(q,\bar{q})) > +$$

$$< p, B(\bar{q}, (e^{2i\theta_0}I - J)^{-1}B(q,q)) > + \frac{e^{-i\theta_0}(1-2e^{i\theta_0})}{1-e^{i\theta_0}} < p, B(q,q) >< p, B(q,\bar{q}) >$$

$$-\frac{2}{1-e^{-i\theta_0}}| < p, B(q,\bar{q}) > |^2 - \frac{e^{i\theta_0}}{e^{3i\theta_0}-1}| < p, B(\bar{q},\bar{q}) > |^2$$

$$C(q,q,\bar{q}) = \begin{pmatrix} \frac{-6(\beta-A+1)e^{-i\theta_0}+2\beta(1+2e^{-2i\theta_0})+2(2+e^{-2i\theta_0})}{(\beta+1)^3} \\ 0 \\ 0 \end{pmatrix}$$

$$< p, C(q,q,\bar{q}) >= \frac{\beta+1}{2(\beta+1)-\beta e^{i\theta_0}+e^{3i\theta_0}} \frac{-6(\beta-A+1)e^{-i\theta_0}+2\beta(1+2e^{-2i\theta_0})+2(2+e^{-2i\theta_0})}{(\beta+1)^3}$$

$$< p, C(q,q,\bar{q}) >= \frac{-6(\beta-A+1)e^{-i\theta_0}+2\beta(1+2e^{-2i\theta_0})+2(2+e^{-2i\theta_0})}{(\beta+1)^2\left(2(\beta+1)-\beta e^{i\theta_0}+e^{3i\theta_0}\right)}$$

To calculate $< p, B(q, (I-J)^{-1}B(q,\bar{q})) >$

$$(I-J)^{-1} = \begin{pmatrix} \frac{1}{\beta+1} & \frac{\beta-A+1}{\beta+1} & \frac{-1}{\beta+1} \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{\beta+1}{\beta-A+1} & \frac{A-\beta}{\beta-A+1} & \frac{1}{\beta-A+1} \\ \frac{\beta+1}{\beta-A+1} & \frac{1}{\beta-A+1} & \frac{1}{\beta-A+1} \\ \frac{\beta+1}{\beta-A+1} & \frac{1}{\beta-A+1} & \frac{\beta-A+2}{\beta-A+1} \end{pmatrix}$$

$$(I-J)^{-1}B(q,\bar{q}) = \begin{pmatrix} \frac{2(\beta-A+1)-2(\beta+1)\cos\theta_0}{(\beta+1)(\beta-A+1)} \\ \frac{2(\beta-A+1)-2(\beta+1)\cos\theta_0}{(\beta+1)(\beta-A+1)} \\ \frac{2(\beta-A+1)-2(\beta+1)\cos\theta_0}{(\beta+1)(\beta-A+1)} \end{pmatrix} = \begin{pmatrix} S \\ S \\ S \end{pmatrix}$$

$$B(q, (I-J)^{-1}B(q,\bar{q})) =$$

$$\begin{pmatrix} M \\ 0 \\ 0 \end{pmatrix}$$

Where

$$M = \frac{2(\beta - A + 1)}{(\beta+1)^2} S e^{-i\theta_0} - \frac{\beta}{(\beta+1)^2} S(1 + e^{-i\theta_0}) - \frac{1}{(\beta+1)^2} S(e^{-i\theta_0} + e^{-2i\theta_0})$$

$$S = \frac{2(\beta - A + 1) - 2(\beta+1)\cos\theta_0}{(\beta+1)(\beta - A + 1)}$$

$$< p, B(q, (I-J)^{-1} B(q, \bar{q})) >= \frac{(\beta+1)M}{2(\beta+1) - \beta e^{i\theta_0} + e^{3i\theta_0}}$$

To find $< p, B(\bar{q}, (e^{2i\theta_0} I - J)^{-1} B(q, q) >$

$$(e^{2i\theta_0} I - J)^{-1} = \begin{pmatrix} e^{2i\theta_0} - \frac{\beta}{\beta+1} & \frac{\beta - A + 1}{\beta+1} & \frac{-1}{\beta+1} \\ -1 & e^{2i\theta_0} & 0 \\ 0 & -1 & e^{2i\theta_0} \end{pmatrix}^{-1}$$

$$= \frac{1}{D} \begin{pmatrix} e^{4i\theta_0} & \frac{1 - (\beta - A + 1) e^{2i\theta_0}}{\beta+1} & \frac{e^{2i\theta_0}}{\beta+1} \\ e^{2i\theta_0} & e^{4i\theta_0} - \frac{\beta}{\beta+1} e^{2i\theta_0} & \frac{1}{\beta+1} \\ 1 & e^{2i\theta_0} - \frac{\beta}{\beta+1} & e^{4i\theta_0} - \frac{\beta}{\beta+1} e^{2i\theta_0} + \frac{\beta - A + 1}{\beta+1} \end{pmatrix}$$

Where $D = \dfrac{(\beta+1)e^{6i\theta_0} - \beta e^{4i\theta_0} + (\beta - A + 1)e^{2i\theta_0} - 1}{\beta + 1}$

$$(e^{2i\theta_0}I-J)^{-1}B(q,q)=\begin{pmatrix} L\frac{e^{4i\theta_0}}{D} \\ \\ L\frac{e^{2i\theta_0}}{D} \\ \\ \frac{L}{D} \\ \end{pmatrix}$$

$$B(\bar{q},(e^{2i\theta_0}I-J)^{-1}B(q,q))$$

$$=\begin{pmatrix} \frac{2(\beta-A+1)L}{(\beta+1)^2D}e^{3i\theta_0}-\frac{\beta L}{(\beta+1)^2D}(e^{2i\theta_0}+e^{5i\theta_0})-\frac{L}{(\beta+1)^2D}(e^{i\theta_0}+e^{4i\theta_0}) \\ \\ 0 \\ \\ 0 \\ \end{pmatrix}$$

$$< p, B(\bar{q},(e^{2i\theta_0}I-J)^{-1}B(q,q) >$$

$$=\frac{L}{D(\beta+1)(2(\beta+1)-\beta e^{i\theta_0}+e^{3i\theta_0})}\Big(2(\beta-A+1)e^{3i\theta_0}-\beta(e^{2i\theta_0}+e^{5i\theta_0})-(e^{i\theta_0}+e^{4i\theta_0})\Big)$$

$$\begin{aligned} A(\beta^*)=\frac{1}{2}\mathbb{R}e\{e^{-i\theta_0}[< p, C(q,q,\bar{q}) > +2 < p, B(q,(I-J)^{-1}B(q,\bar{q})) > \\ + < p, B(\bar{q},(e^{2i\theta_0}I-J)^{-1}B(q,q)) >]\} \end{aligned}$$

Let$R_1=\mathbb{R}e\{e^{-i\theta_0} < p, C(q,q,\bar{q}) >\}$

$$R_1=\mathbb{R}e\{\frac{-6(\beta-A+1)e^{-2i\theta_0}+2\beta(e^{-i\theta_0}+2e^{-3i\theta_0})+2(2e^{-i\theta_0}+e^{-3i\theta_0})}{(\beta+1)^2(2(\beta+1)-\beta e^{i\theta_0}+e^{3i\theta_0})}\}$$

Multipling and dividing by the conjugate of the denominator, the numerator becomes,

$$(4\beta(\beta+1)+8(\beta+1))\,e^{-i\theta_0}+(-12(\beta-A+1)(\beta+1)-4\beta-2\beta^2)\,e^{-2i\theta_0}$$

$$+\,(4(\beta+1)+8\beta(\beta+1)+6\beta(\beta-A+1))\,e^{-3i\theta_0}+(-4\beta^2+4)\,e^{-4i\theta_0}$$

$-6(\beta - A + 1)e^{-5i\theta_0} + (2 + 4\beta)e^{-6i\theta_0}$

Taking the real part of the numerator, and denote it by C_1

$C_1 = (8(\beta + 1) + 4\beta(\beta + 1))\cos\theta_0 + (-12(\beta - A + 1)(\beta + 1) - 4\beta - 2\beta^2)\cos 2\theta_0$

$+ (-4(\beta + 1) + 8\beta(\beta + 1) + 6\beta(\beta - A + 1))\cos 3\theta_0$

$+ (-4\beta^2 + 4)\cos 4\theta_0 - 6(\beta - A + 1)\cos 5\theta_0 + (2 + 4\beta)\cos 6\theta_0$

Multiplying the denominator by its conjugate, we get

$(\beta + 1)^2[4(\beta + 1)^2 - 2\beta(\beta + 1)e^{i\theta_0} - 2\beta(\beta + 1)e^{-i\theta_0} + 2(\beta + 1)e^{3i\theta_0}$

$+ \beta^2 - \beta e^{2i\theta_0} + 2(\beta + 1)e^{-3i\theta_0} - \beta e^{-2i\theta_0} + 1]$

Which is equal,

$C_2 = 4(\beta + 1)^2 + \beta^2 + 1 - 4\beta(\beta + 1)\cos\theta_0 - 2\beta\cos 2\theta_0 + 4(\beta + 1)\cos 3\theta_0$

$R_1 = \mathbb{R}e\{e^{-i\theta_0} < p, C(q, q, \bar{q}) >\} = \dfrac{C_1}{(\beta + 1)^2 C_2}$

Let $R_2 = \mathbb{R}e\{e^{-i\theta_0} < p, B(q, (I - J)^{-1}B(q, \bar{q})) >\} = \mathbb{R}e\{\dfrac{2(\beta + 1)Me^{-i\theta_0}}{2(\beta + 1) - \beta e^{i\theta_0} + e^{3i\theta_0}}\}$

$R_2 = \mathbb{R}e\{\dfrac{4(\beta - A + 1)Se^{-2i\theta_0} - 2\beta S(e^{-i\theta_0} + e^{-2i\theta_0}) - 2S(e^{-3i\theta_0} + e^{-2i\theta_0})}{(\beta + 1)(2(\beta + 1) - \beta e^{i\theta_0} + e^{3i\theta_0})}\}$

Multipling and dividing by the conjugate of the denominator, the numerator becomes,

$8(\beta - A + 1)(\beta + 1)Se^{-2i\theta_0} - 4\beta(\beta + 1)S(e^{-i\theta_0} + e^{-2i\theta_0}) -$

$4(\beta + 1)S(e^{-2i\theta_0} + e^{-3i\theta_0}) - 4\beta(\beta - A + 1)Se^{-3i\theta_0} + 2\beta^2 S(e^{-2i\theta_0} + e^{-3i\theta_0}) +$

$2\beta S(e^{-3i\theta_0} + e^{-4i\theta_0}) + 4(\beta - A + 1)Se^{-5i\theta_0} - 2\beta S(e^{-4i\theta_0} + e^{-5i\theta_0}) - 2S(e^{-5i\theta_0} + e^{-6i\theta_0})$

Taking the real part and denote it by C_3, we get,

$C_3 = -4\beta(\beta+1)S\cos\theta_0 +$

$(8(\beta+1)(\beta-A+1)S - 4\beta(\beta+1)S - 4(\beta+1)S + 2\beta^2 S)\cos 2\theta_0$

$+ (2\beta S + 2\beta^2 S - 4(\beta+1)S - 4\beta S(\beta - A + 1))\cos 3\theta_0 +$

$(4(\beta-A+1)S - 2\beta S - 2S)\cos 5\theta_0 - 2S\cos 6\theta_0$

So $R_2 = \dfrac{C_3}{(\beta+1)C_2}$

Let $R_3 = \mathbb{R}e\{e^{-i\theta_0} < p, B(\bar{q}, (e^{2i\theta_0}I - J)^{-1}B(q,q)) >\}$

$$R_3 = \mathbb{R}e\{\frac{L(2(\beta-A+1)e^{3i\theta_0} - \beta(e^{2i\theta_0} + e^{5i\theta_0}) - (e^{i\theta_0} + e^{4i\theta_0}))e^{-i\theta_0}}{D(\beta+1)(2(\beta+1) - \beta e^{i\theta_0} + e^{3i\theta_0})}\}$$

$$= \mathbb{R}e\{\frac{(2(\beta-A+1)e^{-2i\theta_0} - 2\beta e^{-i\theta_0} - 2e^{-3i\theta_0})(2(\beta-A+1)e^{2i\theta_0} - \beta(e^{i\theta_0} + e^{4i\theta_0}) - (1 + e^{3i\theta_0}))}{((\beta+1)e^{6i\theta_0} - \beta e^{4i\theta_0} + (\beta-A+1)e^{2i\theta_0} - 1)(2(\beta+1) - \beta e^{i\theta_0} + e^{3i\theta_0})}\}$$

The numerator is,

$4(\beta-A+1)^2 - 2\beta(\beta-A+1)(e^{-i\theta_0} + e^{2i\theta_0}) - 2(\beta-A+1)(e^{-2i\theta_0} + e^{i\theta_0})$

$-4\beta(\beta-A+1)e^{i\theta_0} + 2\beta^2(1 + e^{3i\theta_0}) + 2\beta(e^{-i\theta_0} + e^{2i\theta_0})$

$-4(\beta-A+1)e^{-i\theta_0} + 2\beta(e^{-2i\theta_0} + e^{i\theta_0} + 2(e^{-3i\theta_0} + 1)$

Which is equivalent to,

$a_0 + a_1 e^{-i\theta_0} + a_2 e^{i\theta_0} + a_3 e^{-2i\theta_0} + a_4 e^{2i\theta_0} + 2\beta^2 e^{3i\theta_0} + 2e^{-3i\theta_0}$

Where,
$a_0 = 4(\beta-A+1)^2 + 2\beta^2 + 2$
$a_1 = 2\beta - 2\beta(\beta-A+1) - 4(\beta-A+1)$
$a_2 = 2\beta - 2(\beta-A+1) - 4\beta(\beta-A+1)$
$a_3 = 2\beta - 2(\beta-A+1)$

$a_4 = 2\beta - 2\beta(\beta - A + 1)$

The denominator is

$$2(\beta+1)^2 e^{6i\theta_0} - 2\beta(\beta+1)e^{4i\theta_0} + 2(\beta+1)(\beta-A+1)e^{2i\theta_0}$$

$$-2(\beta+1) - \beta(\beta+1)e^{7i\theta_0} + \beta^2 e^{5i\theta_0} - \beta(\beta-A+1)e^{3i\theta_0} + \beta e^{i\theta_0} + (\beta+1)e^{9i\theta_0} - \beta e^{7i\theta_0}$$

$$+ (\beta - A + 1)e^{5i\theta_0} - e^{3i\theta_0}$$

Which is equivalent to,

$$(\beta+1)e^{9i\theta_0} + a_5 e^{7i\theta_0} + a_6 e^{6i\theta_0} + a_7 e^{5i\theta_0} + a_8 e^{4i\theta_0} + a_9 e^{3i\theta_0} + a_{10} e^{2i\theta_0} + \beta e^{i\theta_0} - 2(\beta+1) \tag{4.3.6}$$

Where
$a_5 = -\beta(\beta+1) - \beta$
$a_6 = 2(\beta+1)^2$
$a_7 = \beta^2 + \beta - A + 1$
$a_8 = -2\beta(\beta+1)$
$a_9 = -\beta(\beta - A + 1) - 1$
$a_{10} = 2(\beta - A + 1)(\beta + 1)$

The denominator conjugate is,

$$(\beta+1)e^{-9i\theta_0} + a_5 e^{-7i\theta_0} + a_6 e^{-6i\theta_0} + a_7 e^{-5i\theta_0} + a_8 e^{-4i\theta_0} + a_9 e^{-3i\theta_0} + a_{10} e^{-2i\theta_0} + \beta e^{-i\theta_0} - 2(\beta+1) \tag{4.3.7}$$

Multiply the numerator by (4.3.7) we get,

$$-2(\beta+1)a_0 + a_2\beta + a_4 a_{10} + 2\beta^2 a_9$$

$$+ (\beta a_0 - 2(\beta+1)a_1 + a_2 a_{10} + 2a_8\beta^2 + a_4 a_9)e^{-i\theta_0}$$

$$+ (\beta a_4 - 2(\beta+1)a_2 + 2a_{10}\beta^2)e^{i\theta_0}$$

$$+ (a_0 a_{10} + \beta a_1 + a_2 a_9 + a_4 a_8 - 2(\beta+1)a_3 + 2a_7\beta^2)e^{-2i\theta_0}$$

$$+ (2\beta^3 - 2a_4(\beta+1))e^{2i\theta_0} + (a_0 a_9 + a_1 a_{10} + a_2 a_8 + a_3\beta + a_4 a_7 + 2\beta^2 a_6 - 4(\beta +$$

$1))e^{-3i\theta_0}$

$-4\beta^2(\beta+1)e^{3i\theta_0} + (a_0a_8 + a_1a_9 + a_2a_7 + a_3a_{10} + a_4a_6 + 2a_5\beta^2 + 2\beta)e^{-4i\theta_0}$

$+ (a_0a_7 + a_1a_8 + a_2a_6 + a_3a_9 + a_4a_5 + 2a_{10})e^{-5i\theta_0}$

$+ (a_0a_6 + a_1a_7 + a_2a_5 + a_3a_8 + 2\beta^2(\beta+1) + 2a_9)e^{-6i\theta_0} +$

$(a_0a_5 + a_1a_6 + a_3a_7 + a_4(\beta+1) + 2a_8)e^{-7i\theta_0} + (a_1a_5 + a_2(\beta+1) + a_3a_6 + 2a_7)e^{-8i\theta_0}$

$+ (a_0(\beta+1) + a_3a_5 + 2a_6)e^{-9i\theta_0} + (a_1(\beta+1) + 2a_5)e^{-10i\theta_0}$

$+ a_3(\beta+1)e^{-11i\theta_0} + 2(\beta+1)e^{-12i\theta_0}$

Taking the real part of the previous expression, and denote it by C_4.

$C_4 = -2(\beta+1)a_0 + a_2\beta + a_4a_{10} + 2\beta^2 a_9$

$+ (\beta a_0 - 2(\beta+1)a_1 + a_2a_{10} + a_4a_9 + 2a_8\beta^2 + \beta a_4 - 2(\beta+1)a_2 + 2a_{10}\beta^2)\cos\theta_0$

$+ (a_0a_{10} + \beta a_1 + a_2a_9 + a_4a_8 - 2(\beta+1)a_3 + 2a_7\beta^2 + 2\beta^3 - 2a_4(\beta+1))\cos 2\theta_0$

$+ (a_0a_9 + a_1a_{10} + a_2a_8 + a_3\beta + a_4a_7 + 2\beta^2 a_6 - 4(\beta+1) - 4\beta^2(\beta+1))\cos 3\theta_0$

$+ (a_0a_8 + a_1a_9 + a_2a_7 + a_3a_{10} + a_4a_6 + 2a_5\beta^2 + 2\beta)\cos 4\theta_0$

$+ (a_0a_7 + a_1a_8 + a_2a_6 + a_3a_9 + a_4a_5 + 2a_{10})\cos 5\theta_0$

$+ (a_0a_6 + a_1a_7 + a_2a_5 + a_3a_8 + 2\beta^2(\beta+1) + 2a_9)\cos 6\theta_0$

$+ (a_0a_5 + a_1a_6 + a_3a_7 + a_4(\beta+1) + 2a_8)\cos 7\theta_0$

$+ (a_1a_5 + a_2(\beta+1) + a_3a_6 + 2a_7)\cos 8\theta_0 + (a_0(\beta+1) + a_3a_5 + 2a_6)\cos 9\theta_0$

$+ (a_1(\beta+1) + 2a_5)\cos 10\theta_0 + a_3(\beta+1)\cos 11\theta_0 + 2(\beta+1)\cos 12\theta_0$

Now multiply (4.3.6) by (4.3.7), and denote it by C_5,

$C_5 = 5(\beta+1)^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2 + a_9^2 + a_{10}^2 + \beta^2 +$

$(a_5a_6 + a_6a_7 + a_7a_8 + a_8a_9 + a_9a_{10} + a_{10}\beta - 2\beta(\beta+1))e^{i\theta_0} +$

$(a_5a_6 + a_6a_7 + a_7a_8 + a_8a_9 + a_9a_{10} + a_{10}\beta - 2\beta(\beta+1))e^{-i\theta_0} +$

$((\beta+1)a_5 + a_5a_7 + a_6a_8 + a_7a_9 + a_8a_{10} + a_9\beta - 2a_{10}(\beta+1))e^{2i\theta_0} +$

$((\beta+1)a_5 + a_5a_7 + a_6a_8 + a_7a_9 + a_8a_{10} + a_9\beta - 2a_{10}(\beta+1))e^{-2i\theta_0} +$

$((\beta+1)a_6 + a_5a_8 + a_6a_9 + a_7a_{10} + a_8\beta - 2a_9(\beta+1))e^{3i\theta_0} +$

$((\beta+1)a_6 + a_5a_8 + a_6a_9 + a_7a_{10} + a_8\beta - 2a_9(\beta+1))e^{-3i\theta_0} +$

$((\beta+1)a_7 + a_5a_9 + a_6a_{10} + a_7\beta - 2a_8(\beta+1))e^{4i\theta_0} +$

$((\beta+1)a_7 + a_5a_9 + a_6a_{10} + a_7\beta - 2a_8(\beta+1))e^{-4i\theta_0} +$

$((\beta+1)a_8 + a_5a_{10} + a_6\beta - 2a_7(\beta+1))e^{5i\theta_0} +$
$((\beta+1)a_8 + a_5a_{10} + a_6\beta - 2a_7(\beta+1))e^{-5i\theta_0} +$

$((\beta+1)a_9 + a_5\beta - 2a_6(\beta+1))e^{6i\theta_0} +$

$((\beta+1)a_9 + a_5\beta - 2a_6(\beta+1))e^{-6i\theta_0} +$

$((\beta+1)a_{10} - 2a_5(\beta+1))e^{7i\theta_0} + ((\beta+1)a_{10} - 2a_5(\beta+1))e^{-7i\theta_0} +$

$\beta(\beta+1)e^{8i\theta_0} + \beta(\beta+1)e^{-8i\theta_0} - 2(\beta+1)^2e^{9i\theta_0} - 2(\beta+1)^2e^{-9i\theta_0}$

Which is equal to,
$C_5 = 5(\beta+1)^2 + a_5^2 + a_6^2 + a_7^2 + a_8^2 + a_9^2 + a_{10}^2 + \beta^2 +$

$2(a_5a_6 + a_6a_7 + a_7a_8 + a_8a_9 + a_9a_{10} + a_{10}\beta - 2\beta(\beta+1))\cos\theta_0 +$

$2((\beta+1)a_5 + a_5a_7 + a_6a_8 + a_7a_9 + a_8a_{10} + a_9\beta - 2a_{10}(\beta+1))\cos 2\theta_0 +$

$2((\beta+1)a_6 + a_5a_8 + a_6a_9 + a_7a_{10} + a_8\beta - 2a_9(\beta+1))\cos 3\theta_0 +$

$2((\beta+1)a_7 + a_5a_9 + a_6a_{10} + a_7\beta - 2a_8(\beta+1))\cos 4\theta_0 +$

$2((\beta+1)a_8 + a_5a_{10} + a_6\beta - 2a_7(\beta+1))\cos 5\theta_0 +$

$2((\beta+1)a_9 + a_5\beta - 2a_6(\beta+1))\cos 6\theta_0 + 2((\beta+1)a_{10} - 2a_5(\beta+1))\cos 7\theta_0 +$

$2\beta(\beta+1)\cos 8\theta_0 - 4(\beta+1)^2\cos 9\theta_0$

Where,
$\cos\theta_0 = \frac{-A}{2}$

$\cos 2\theta_0 = 2\cos^2\theta_0 - 1 = \frac{A^2}{2} - 1$

$\cos 3\theta_0 = 4\cos^3\theta_0 - 3\cos\theta_0 = \frac{3A-A^3}{2}$

$\cos 4\theta_0 = 2\cos^2 2\theta_0 - 1 = 2(\frac{A^2}{2} - 1)^2 - 1$

$\cos 5\theta_0 = 2\cos 2\theta_0\cos 3\theta_0 - \cos\theta = (\frac{A^2}{2} - 1)(3A - A^3) + \frac{A}{2}$

$\cos 6\theta_0 = 2\cos^2 3\theta_0 - 1 = 2(\frac{3A}{2} - \frac{A^3}{2})^2 - 1$

$\cos 7\theta_0 = 2\cos 2\theta_0\cos 5\theta_0 - \cos 3\theta_0$
$= 2(\frac{A^2}{2} - 1)\left((\frac{A^2}{2} - 1)(3A - A^3) + \frac{A}{2}\right) + \frac{3A-A^3}{2}$

$\cos 8\theta_0 = 2\cos^2 4\theta_0 - 1 = 2\left(2(\frac{A^2}{2} - 1)^2 - 1\right)^2 - 1$

$\cos 9\theta_0 = 2\cos 4\theta_0\cos 5\theta_0 - \cos\theta_0$
$\cos 9\theta_0 = 2\left(2(\frac{A^2}{2} - 1)^2 - 1\right)\left((\frac{A^2}{2} - 1)(3A - A^3) + \frac{A}{2}\right) + \frac{A}{2}$

$\cos 10\theta_0 = 2\cos^2 5\theta_0 - 1 = 2\left((\frac{A^2}{2} - 1)(3A - A^3) + \frac{A}{2}\right)^2 - 1$

$\cos 11\theta_0 = 2\cos 5\theta_0\cos 6\theta_0 - \cos\theta_0$
$= 2\left((\frac{A^2}{2} - 1)(3A - A^3) + \frac{A}{2}\right)\left(2(\frac{3A}{2} - \frac{A^3}{2})^2 - 1\right) + \frac{A}{2}$

$\cos 12\theta_0 = 2\cos^2 6\theta_0 - 1 = 2\left(2(\frac{3A}{2} - \frac{A^3}{2})^2 - 1\right)^2 - 1$

$$A(\beta^*) = \frac{1}{2}R_1 + R_2 + \frac{1}{2}R_3$$

$$A(\beta^*) = \frac{1}{2}\left(\frac{C_1}{(\beta+1)^2C_2}\right) + \frac{C_3}{(\beta+1)C_2} + \frac{1}{2}\left(\frac{C_4}{C_5}\right)$$

$$= \frac{C_1 + 2(\beta+1)C_3}{2(\beta+1)^2C_2} + \frac{C_4}{2C_5}$$

Theorem 4.3.3. *If $A(\beta^*) < 0$(respectively, > 0), then the Neimark- Sacker bifurcation at $\beta = \beta^*$ is supercritical(respectively, subcritical) and there exists a unique invariant closed curve that bifurcates from the fixed point which is asymptotically stable (respectively, unstable).*

5. Fourth order rational difference equation

5.1 Introduction

Camouzis [4] gave an analytical description of the local stability of the positive equilibrium point of

$$X_{n+1} = \frac{\sigma X_{n-2} + X_{n-3}}{A + X_{n-3}} \tag{5.1.1}$$

with positive parameters σ and A and non negative initial conditions, also he investigated the global attractivity of the positive fixed point, and derived the following results:

1. The positive fixed point is locally stable when

$$\sigma^3 + \sigma^2 - (2A^2 + 4A + 2)\sigma + A^3 + A^2 - A - 1 < 0$$

2. The positive fixed point is locally unstable when

$$\sigma^3 + \sigma^2 - (2A^2 + 4A + 2)\sigma + A^3 + A^2 - A - 1 > 0$$

3. Assume that $A-1 < \sigma < A+1$ then every positive solution of equation 5.1 converges to the positive equilibrium point.

R.Zahang and X.Ding [6] studied the existence and direction of Neimark Sacker bifurcation of the same equation and gave the following results:

Theorem 5.1.1. *Suppose $\sigma > A - 1$ when σ satisfies $\sigma^3 + \sigma^2 - (2A^2 + 4A + 2)\sigma + A^3 + A^2 - A - 1 < 0$ then Neimark Sacker bifurcation occurs.*

Theorem 5.1.2. *If $a(\sigma^*) < 0$ (respectively > 0) then the Neimark Sacker bifurcation is supercritical (respectively subcritical) and unique closed invariant curve bifurcating from the positive fixed point is asymptotically stable (respectively unstable).*

5.2 Dynamics and Bifurcation of the fourth order equation

Consider the equation

$$X_{n+1} = \frac{\beta X_n + X_{n-3}}{A + X_{n-1}} \tag{5.2.1}$$

with positive parameters and initial conditions. To find the fixed points we solve the equation $f(x, x, x, x) = x$
So

$$X^* = \frac{(\beta+1)X^*}{A + X^*}$$

There are two fixed points, the zero fixed point where $X^* = (0, 0, 0, 0)$ and $X^* = (\beta - A + 1, \beta - A + 1, \beta - A + 1, \beta - A + 1)$. We assume that $\beta + 1 > A$

Let $\begin{pmatrix} U_n \\ V_n \\ W_n \\ Z_n \end{pmatrix} = \begin{pmatrix} X_n \\ X_{n-1} \\ X_{n-2} \\ X_{n-3} \end{pmatrix}$ then

$$\begin{pmatrix} U_{n+1} \\ V_{n+1} \\ W_{n+1} \\ Z_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\beta U_n + Z_n}{A + V_n} \\ U_n \\ V_n \\ W_n \end{pmatrix} \tag{5.2.2}$$

The Jacobian matrix of (5.2.2) at the positive fixed point is

$$J = \begin{pmatrix} \frac{\beta}{A+X^*} & \frac{-(\beta X^* + X^*)}{(A+X^*)^2} & 0 & \frac{1}{A+X^*} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$J = \begin{pmatrix} \frac{\beta}{\beta+1} & \frac{-(\beta - A + 1)}{\beta+1} & 0 & \frac{1}{\beta+1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

The characteristic equation of the Jacobian matrix is $P(\lambda) = |J - \lambda I| = 0$

$$P(\lambda) = \begin{vmatrix} \frac{\beta}{\beta+1} - \lambda & \frac{-(\beta-A+1)}{\beta+1} & 0 & \frac{1}{\beta+1} \\ 1 & -\lambda & 0 & 0 \\ 0 & 1 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$P(\lambda) = \left(\frac{\beta}{\beta+1} - \lambda\right)(-\lambda)^3 + \frac{\beta - A + 1}{\beta+1}\lambda^2 - \frac{1}{\beta+1}$$

$$P(\lambda) = \lambda^4 - \frac{\beta}{\beta+1}\lambda^3 + \frac{\beta - A + 1}{\beta+1}\lambda^2 - \frac{1}{\beta+1} \tag{5.2.3}$$

Theorem 5.2.1. *The positive fixed point is asymptotically stable if* $A > \frac{4(\beta+1)}{(\beta+2)^2}$ *and unstable if* $A < \frac{4(\beta+1)}{(\beta+2)^2}$

Proof. To study the stability of the fixed point we use the next theorem

Theorem 5.2.2. *[4] For a polynomial* $F(\lambda) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ *to have roots in the unit circle, the following conditions must be satisfied*

$$|a_1 + a_3| < 1 + a_0 + a_2, \qquad |a_1 - a_3| < 2(1 - a_0)$$

$$a_2 - 3a_0 < 3, \qquad a_0 + a_2 + a_0^2 + a_1^2 + a_0^2 a_2 + a_0 a_3^2 < 1 + 2a_0 a_2 + a_1 a_3 + a_0 a_1 a_3 + a_0^3$$

For the polynomial

$$P(\lambda) = \lambda^4 - \frac{\beta}{\beta+1}\lambda^3 + \frac{\beta - A + 1}{\beta+1}\lambda^2 - \frac{1}{\beta+1}$$

$a_0 = \frac{-1}{\beta+1}$, $a_1 = 0$, $a_2 = \frac{\beta-A+1}{\beta+1}$, $a_3 = \frac{-\beta}{\beta+1}$
The first condition

$$|a_1 + a_3| < 1 + a_0 + a_2$$

$$\left|-\frac{\beta}{\beta+1}\right| < 1 - \frac{1}{\beta+1} + \frac{\beta - A + 1}{\beta+1}$$

$$\frac{\beta}{\beta+1} < 1 - \frac{1}{\beta+1} + \frac{\beta - A + 1}{\beta+1}$$

$$\frac{\beta + 1 - \beta + A - 1}{\beta+1} < 1$$

$$\frac{A}{\beta+1} < 1$$

So

$$A < \beta + 1$$

Which is equivalent to $\beta - A + 1 > 0$ and the last inequality is satisfied by assumption.
The second condition is $|a_1 - a_3| < 2(1 - a_0)$

$$\left|\frac{\beta}{\beta+1}\right| < 2(1 + \frac{1}{\beta+1})$$

$$\frac{\beta}{\beta+1} < 2 + \frac{2}{\beta+1}$$

$$\frac{\beta-2}{\beta+1} < 2$$

$$\beta - 2 < 2\beta + 2$$

Thus,

$$\beta + 4 > 0$$

Which is also satisfied for every β.
The third condition is $a_2 - 3a_0 < 3$

$$\frac{\beta - A + 1}{\beta+1} - 3(\frac{-1}{\beta+1}) < 3$$

$$\frac{\beta - A + 4}{\beta+1} < 3$$

$$\beta - A + 4 < 3\beta + 3$$

Thus,

$$2\beta + A - 1 > 0$$

The last inequality holds for every β, since

$$1 - 2\beta < A < \beta + 1$$

The fourth condition is

$$a_0 + a_2 + a_0^2 + a_1^2 + a_0^2 a_2 + a_0 a_3^2 < 1 + 2a_0 a_2 + a_1 a_3 + a_0 a_1 a_3 + a_0^3$$

Which gives,

$$\frac{-1}{\beta+1}+\frac{\beta-A+1}{\beta+1}+\left(\frac{1}{\beta+1}\right)^2\left(1+\frac{\beta-A+1}{\beta+1}\right)+\frac{-1}{\beta+1}\left(\frac{-\beta}{\beta+1}\right)^2$$

$$<1+\left(\frac{-2}{\beta+1}\right)\left(\frac{\beta-A+1}{\beta+1}\right)+\left(\frac{-1}{\beta+1}\right)^3$$

$$\left(\frac{-1}{\beta+1}\right)\left(1+\frac{\beta^2}{(\beta+1)^2}\right)+\frac{1}{(\beta+1)^2}\left(1+\frac{\beta-A+1}{\beta+1}\right)+\frac{\beta-A+1}{\beta+1}$$

$$<1-\frac{2(\beta-A+1)}{(\beta+1)^2}-\frac{1}{(\beta+1)^3}$$

Multiply by $(\beta+1)^3$

$$-\left((\beta+1)^2+\beta^2\right)+(\beta+1)+(\beta-A+1)+(\beta-A+1)(\beta+1)^2<(\beta+1)^3-2(\beta-A+1)(\beta+1)-1$$

$$-(2\beta^2+2\beta+1)+2\beta-A+2+(\beta-A+1)(\beta+1)(\beta+3)<(\beta+1)^3-1$$

$$-2\beta^2-A+1+(\beta-A+1)(\beta+1)(\beta+3)<(\beta+1)^3-1$$

$$-2\beta^2-A+2+(\beta-A+1)(\beta+1)(\beta+3)<\beta^3+3\beta^2+3\beta+1$$

$$-2\beta^2-A+1+(\beta-A+1)(\beta+1)(\beta+3)<\beta^3+3\beta^2+3\beta$$

$$1-A+(\beta+1)^2(\beta+3)-A(\beta+1)(\beta+3)<\beta^3+5\beta^2+3\beta$$

$$1-A(1+(\beta+1)(\beta+3))+(\beta+1)^2(\beta+3)<\beta^3+5\beta^2+3\beta$$

$$1-A(1+\beta^2+4\beta+3)+\beta^3+5\beta^2+7\beta+3<\beta^3+5\beta^2+3\beta$$

$$1-A(\beta^2+4\beta+4)<-4\beta-3$$

$$-A(\beta+2)^2<-4\beta-4$$

$$A>4\frac{(\beta+1)}{(\beta+2)^2}=A^*. \quad A^*<1.$$

For this condition on A the eigenvalues of the characteristic equation will lie within the unit circle, hence the fixed point is stable. □

5.3 Direction and Stability of Neimark Sacker bifurcation

Theorem 5.3.1. *If* $A = A^* = \frac{4(\beta+1)}{(\beta+2)^2}$ *then (5.2.3) has two complex conjugate roots that lie on the unit circle . Moreover the Neimark Sacker bifurcation conditions are satisfied.*

Proof. First we show that equation (5.2.3) has two complex conjugate roots, using Descartes and $Vi\grave{e}te$ theorem.

Theorem 5.3.2. *(Descartes theorem)[2] The number of positive roots (counted considering their multiplicity) of a polynomial* $P_n(x)$ *with real coefficients is either equal to the number of sign alterations between consecutive nonzero coefficients or is less than it by a multiple of* 2.

Applying the Descartes theorem to $P_n(-x)$, we obtain a similar theorem for the negative roots of the polynomial $P_n(x)$. So the number of negative roots of a polynomial $P_n(x)$ is equal to the number of positive roots of the polynomial $P_n(-x)$.

Theorem 5.3.3. *(Viète theorem)[1] Let* $\alpha, \sigma, \gamma, \delta$ *be the roots of the polynomial*

$$P(x) = ax^4 + bx^3 + cx^2 + dx + e = 0$$

then

$$\alpha + \sigma + \gamma + \delta = \frac{-b}{a}$$
$$\alpha\sigma + \sigma\gamma + \gamma\delta + \alpha\gamma + \alpha\delta + \sigma\delta = \frac{c}{a}$$
$$\alpha\sigma\gamma + \alpha\gamma\delta + \alpha\sigma\delta + \sigma\gamma\delta = \frac{-d}{a}$$
$$\alpha\sigma\gamma\delta = \frac{e}{a}$$

Applying Descartes theorem to (5.2.3), the alteration in sign is $(+-+-)$ so it has one positive root or three positive roots. Also applying Descartes theorem to $P(-\lambda)$, the alteration of sign is $(+++-)$ so $P(\lambda)$ has one negative root.

$$P(0) = \frac{-1}{\beta+1} < 0$$

$$P(1) = 1 - \frac{\beta}{\beta+1} + \frac{\beta - A + 1}{\beta+1} - \frac{1}{\beta+1} = \frac{\beta - A + 1}{\beta+1} > 0$$

$$P(-1) = 1 + \frac{\beta}{\beta+1} + \frac{\beta - A + 1}{\beta+1} - \frac{1}{\beta+1} = \frac{3\beta - A + 1}{\beta+1} > 0$$

So there exists two real roots say $\mu_1 \in (0,1)$ and $\mu_2 \in (-1,0)$.
Also

$$P'(\lambda) = 4\lambda^3 - \frac{3\beta}{\beta+1}\lambda^2 + \frac{2(\beta - A + 1)}{\beta+1}\lambda = 0$$

$$\lambda\left(4\lambda^2 - \frac{3\beta}{\beta+1}\lambda + \frac{2(\beta - A + 1)}{\beta+1}\right) = 0$$

So $\lambda = 0$ or $4\lambda^2 - \frac{3\beta}{\beta+1}\lambda + \frac{2(\beta-A+1)}{\beta+1} = 0$

equivelantly $4\lambda^2(\beta+1) - 3\beta\lambda + 2(\beta - A + 1) = 0$
Which gives,

$$\lambda = \frac{3\beta \pm \sqrt{9\beta^2 - 32(\beta - A + 1)(\beta+1)}}{8(\beta+1)}$$

But the discriminant of the previous quadratic equation is negative since,

$$\Delta = 9\beta^2 - 32(\beta - A + 1)(\beta + 1)$$

$$= 9\beta^2 - 32(\beta^2 - \beta A + 2\beta - A + 1)$$

$$= -23\beta^2 + 32\beta A - 64\beta + 32A - 32$$

$$= -23\beta^2 - 64\beta - 32 + 32A(\beta+1)$$

$$< -23\beta^2 - 64\beta - 32 + 32(\beta+1) = -23\beta^2 - 32\beta < 0$$

So $P'(\lambda)$ has one real root, hence $P(\lambda)$ changes its direction only once. To show that the positive real root is simple, by the way of contradiction suppose it has multiplicity equal three, then by *Viète* theorem,

$$3\mu_1 + \mu_2 = \frac{\beta}{\beta+1} \tag{5.3.1}$$

$$3\mu_1\mu_2 + 3\mu_1^2 = \frac{\beta - A + 1}{\beta+1} \tag{5.3.2}$$

$$3\mu_2\mu_1^2 + \mu_1^3 = 0 \tag{5.3.3}$$

$$\mu_1^3\mu_2 = \frac{-1}{\beta+1} \tag{5.3.4}$$

From equation (5.3.3) $\mu_1 = -3\mu_2$, substitute in (5.3.4) we get,

$$-27\mu_2^4 = \frac{-1}{\beta+1}$$

$$\mu_2 = \sqrt[4]{\frac{1}{27(\beta+1)}} > 0$$

A contradiction. So equation (5.2.3) has two real roots and two conjugate complex roots.

The next step is to show that $|\lambda_{1,2}| = 1$. We will use *Viète* theorem Let $\mu_{1,2}$ be the real roots of (5.2.3), and $\lambda_2 = \bar{\lambda}_1$

$$\mu_1 + \mu_2 + \lambda_1 + \lambda_2 = \frac{\beta}{\beta+1} \tag{5.3.5}$$

$$\mu_1\mu_2 + \mu_2\lambda_1 + \lambda_1\lambda_2 + \mu_1\lambda_1 + \mu_1\lambda_2 + \mu_2\lambda_2 = \frac{\beta - A + 1}{\beta+1} \tag{5.3.6}$$

$$\mu_1\mu_2\lambda_1 + \mu_1\lambda_1\lambda_2 + \mu_1\mu_2\lambda_2 + \mu_2\lambda_1\lambda_2 = 0 \tag{5.3.7}$$

$$\mu_1\mu_2\lambda_1\lambda_2 = \frac{-1}{\beta+1} \tag{5.3.8}$$

From equation (5.3.8)

$$\mu_1\mu_2\lambda_1\lambda_2 = \mu_1\mu_2 = \frac{-1}{\beta+1} \tag{5.3.9}$$

Substitute in equation (5.3.7)

$$\frac{-1}{\beta+1}\lambda_1 + \mu_1 + \frac{-1}{\beta+1}\lambda_2 + \mu_2 = 0$$

$$\frac{-1}{\beta+1}(\lambda_1 + \lambda_2) + \mu_1 + \mu_2 = 0$$

$$\frac{1}{\beta+1}(\lambda_1 + \lambda_2) = \mu_1 + \mu_2 \tag{5.3.10}$$

Substitute in equation (5.3.5)

$$\frac{1}{\beta+1}(\lambda_1+\lambda_2)+\lambda_1+\lambda_2=\frac{\beta}{\beta+1}$$

$$\left(\frac{1}{\beta+1}+1\right)(\lambda_1+\lambda_2)=\frac{\beta}{\beta+1}$$

$$\frac{\beta+2}{\beta+1}(\lambda_1+\lambda_2)=\frac{\beta}{\beta+1}$$

$$\lambda_1+\lambda_2=\frac{\beta}{\beta+2} \tag{5.3.11}$$

Using equations (5.3.9), (5.3.10) and (5.3.6)

$$\frac{-1}{\beta+1}+\mu_2\lambda_1+1+\mu_1\lambda_1+\mu_1\lambda_2+\mu_2\lambda_2=\frac{\beta-A+1}{\beta+1}$$

$$\frac{\beta}{\beta+1}+\mu_2(\lambda_1+\lambda_2)+\mu_1(\lambda_1+\lambda_2)=\frac{\beta-A+1}{\beta+1}$$

$$(\lambda_1+\lambda_2)(\mu_1+\mu_2)=\frac{\beta-A+1}{\beta+1}-\frac{\beta}{\beta+1}=\frac{1-A}{\beta+1}$$

Using (5.3.10), the last equation gives

$$\frac{1}{\beta+1}(\lambda_1+\lambda_2)^2=\frac{1-A}{\beta+1}$$

$$(\lambda_1+\lambda_2)^2=1-A$$

Using equation (5.3.11) we get

$$\left(\frac{\beta}{\beta+2}\right)^2=1-A$$

$$A=1-\left(\frac{\beta}{\beta+2}\right)^2=\frac{(\beta+2)^2-\beta^2}{(\beta+2)^2}=\frac{4(\beta+1)}{(\beta+2)^2}=A^*$$

Since the roots are uniquely determined, the above argument implies the

existence of conjugate pair of complex roots on the unit circle.
Let $\lambda = e^{i\theta}$

$$P(\lambda) = \lambda^4 - \frac{\beta}{\beta+1}\lambda^3 + \frac{\beta - A + 1}{\beta+1}\lambda^2 - \frac{1}{\beta+1}$$

then

$$P(e^{i\theta}) = e^{4i\theta} - \frac{\beta}{\beta+1}e^{3i\theta} + \frac{\beta - A + 1}{\beta+1}e^{2i\theta} - \frac{1}{\beta+1} = 0$$

$$\cos 4\theta + i\sin 4\theta - \frac{\beta}{\beta+1}(\cos 3\theta + i\sin 3\theta) + \frac{\beta - A + 1}{\beta+1}(\cos 2\theta + i\sin 2\theta) - \frac{1}{\beta+1} = 0$$

Separate the real and imaginary parts

$$\cos 4\theta - \frac{\beta}{\beta+1}\cos 3\theta + \frac{\beta - A + 1}{\beta+1}\cos 2\theta - \frac{1}{\beta+1} = 0$$

$$\sin 4\theta - \frac{\beta}{\beta+1}\sin 3\theta + \frac{\beta - A + 1}{\beta+1}\sin 2\theta = 0$$

Rewrite these equations in the form

$$\cos 4\theta - \frac{\beta}{\beta+1}\cos 3\theta = -\frac{\beta - A + 1}{\beta+1}\cos 2\theta + \frac{1}{\beta+1}$$

$$\sin 4\theta - \frac{\beta}{\beta+1}\sin 3\theta = -\frac{\beta - A + 1}{\beta+1}\sin 2\theta$$

Square both sides of previous equations,

$$\cos^2 4\theta - \frac{2\beta}{\beta+1}\cos 4\theta\cos 3\theta + \left(\frac{\beta}{\beta+1}\right)^2\cos^2 3\theta =$$

$$\left(\frac{1}{\beta+1}\right)^2 - 2\frac{(\beta - A + 1)}{(\beta+1)^2}\cos 2\theta + \left(\frac{\beta - A + 1}{\beta+1}\right)^2\cos^2 2\theta$$

$$\sin^2 4\theta - \frac{2\beta}{\beta+1}\sin 4\theta\sin 3\theta + \left(\frac{\beta}{\beta+1}\right)^2\sin^2 3\theta = \left(\frac{\beta - A + 1}{\beta+1}\right)\sin^2 2\theta$$

Adding the equations up, we get.

$$1-\frac{2\beta}{\beta+1}(\cos 4\theta\cos 3\theta+\sin 4\theta\sin 3\theta)+\left(\frac{\beta}{\beta+1}\right)^2$$
$$=\frac{1}{(\beta+1)^2}+\left(\frac{\beta-A+1}{\beta+1}\right)^2-2\frac{(\beta-A+1)}{(\beta+1)^2}\cos 2\theta$$

$$1-\frac{2\beta}{\beta+1}\cos\theta+\left(\frac{\beta}{\beta+1}\right)^2=\frac{1}{(\beta+1)^2}+\left(\frac{\beta-A+1}{\beta+1}\right)^2-2\frac{\beta-A+1}{(\beta+1)^2}(2\cos^2\theta-1)$$

$$1-\frac{2\beta}{\beta+1}\cos\theta+\frac{\beta^2-1}{(\beta+1)^2}-\left(\frac{\beta-A+1}{\beta+1}\right)^2+4\frac{(\beta-A+1)}{(\beta+1)^2}\cos^2\theta-2\frac{(\beta-A+1)}{(\beta+1)^2}=0$$

$$4\frac{(\beta-A+1)}{(\beta+1)^2}\cos^2\theta-\frac{2\beta}{\beta+1}\cos\theta+1+\frac{\beta-1}{\beta+1}-\left(\frac{\beta-A+1}{\beta+1}\right)\left(\frac{\beta-A+1}{\beta+1}+\frac{2}{\beta+1}\right)=0$$

$$4\frac{(\beta-A+1)}{\beta+1}\cos^2\theta-2\beta\cos\theta+2\beta-\left(\frac{\beta-A+1}{\beta+1}\right)(\beta-A+3)=0$$
$$\cos^2\theta-\frac{2\beta(\beta+1)}{4(\beta-A+1)}\cos\theta+\frac{2\beta(\beta+1)}{4(\beta-A+1)}-\frac{\beta-A+3}{4}=0 \qquad (5.3.12)$$

From equation (5.3.11)

$$\lambda_1+\lambda_2=\frac{\beta}{\beta+2}$$
$$2\cos\theta=\frac{\beta}{\beta+2}$$

so

$$\cos\theta=\frac{\beta}{2(\beta+2)}$$

Note that this is a root of equation 5.3.12,

At $A=\frac{4(\beta+1)}{(\beta+2)^2}$, define $\cos\theta_0=\frac{\beta}{2(\beta+2)}$.
$0<\cos\theta_0<\frac{1}{2}$, hence $\theta_0=\cos^{-1}\left(\frac{\beta}{2(\beta+2)}\right)$ and $\theta_0\in(0,\frac{\pi}{2})$.

Where $\theta_0 \neq 0, \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pi$, it follows that $e^{ik\theta_0} \neq 1$ for $k \in \{1, 2, 3, 4\}$

For the transversality condition we show that $\frac{d|\lambda|^2}{dA}|_{A^*,\theta_0} \neq 0$

$$\frac{d|\lambda|^2}{dA} = \lambda\left(\frac{\partial P(\bar{\lambda})}{\partial A}. \quad \frac{\partial\bar{\lambda}}{\partial P(\bar{\lambda})}\right) + \bar{\lambda}\left(\frac{\partial P(\lambda)}{\partial A}. \quad \frac{\partial\lambda}{\partial P(\lambda)}\right)$$

$$= \lambda\left(\frac{-\bar{\lambda}^2}{\beta+1}. \quad \frac{1}{4\bar{\lambda}^3 - \frac{3\beta}{\beta+1}\bar{\lambda}^2 + \frac{2(\beta-A+1)}{\beta+1}\bar{\lambda}}\right) + \bar{\lambda}\left(\frac{-(\lambda)^2}{\beta+1}. \quad \frac{1}{4\lambda^3 - \frac{3\beta}{\beta+1}\lambda^2 + \frac{2(\beta-A+1)}{\beta+1}\lambda}\right)$$

$$= \frac{-\bar{\lambda}}{(\beta+1)(4\bar{\lambda}^3 - \frac{3\beta}{\beta+1}\bar{\lambda}^2 + \frac{2(\beta-A+1)}{\beta+1}\bar{\lambda})} + \frac{-\lambda}{(\beta+1)(4\lambda^3 - \frac{3\beta}{\beta+1}\lambda^2 + \frac{2(\beta-A+1)}{\beta+1}\lambda)}$$

$$= \frac{-\bar{\lambda}}{4(\beta+1)\bar{\lambda}^3 - 3\beta\bar{\lambda}^2 + 2(\beta-A+1)\bar{\lambda}} + \frac{-\lambda}{4(\beta+1)\lambda^3 - 3\beta\lambda^2 + 2(\beta-A+1)\lambda}$$

$$= \frac{-\bar{\lambda}\left[4(\beta+1)\lambda^3 - 3\beta\lambda^2 + 2(\beta-A+1)\lambda\right] + (-\lambda)\left[4(\beta+1)\bar{\lambda}^3 - 3\beta\bar{\lambda}^2 + 2(\beta-A+1)\bar{\lambda}\right]}{\left[4(\beta+1)(\bar{\lambda})^3 - 3\beta(\bar{\lambda})^2 + 2(\beta-A+1)\bar{\lambda}\right]\left[4(\beta+1)(\lambda)^3 - 3\beta(\lambda)^2 + 2(\beta-A+1)\lambda\right]}$$

$$= \frac{-4(\beta+1)\lambda^2 + 3\beta\lambda - 2(\beta-A+1) - 4(\beta+1)\bar{\lambda}^2 + 3\beta\bar{\lambda} - 2(\beta-A+1)}{\left[4(\beta+1)(\bar{\lambda})^3 - 3\beta(\bar{\lambda})^2 + 2(\beta-A+1)\bar{\lambda}\right]\left[4(\beta+1)(\lambda)^3 - 3\beta(\lambda)^2 + 2(\beta-A+1)\lambda\right]}$$

$$= \frac{-4(\beta+1)(\lambda^2 + \bar{\lambda}^2) + 3\beta(\lambda + \bar{\lambda}) - 4(\beta-A+1)}{L}$$

Where

$$L = 16(\beta+1)^2 + 9\beta^2 + 4(\beta-A+1)^2 - (12\beta(\beta+1) + 6\beta(\beta-A+1))(\bar{\lambda}+\lambda) + 8(\beta-A+1)(\beta+1)(\bar{\lambda}^2+\lambda^2)$$

$$L = 16(\beta+1)^2 + 9\beta^2 + 4(\beta-A^*+1)^2 - (12\beta(\beta+1) + 6\beta(\beta-A^*+1))(2\cos\theta_0) +$$
$$8(\beta-A^*+1)(\beta+1)(2\cos^2\theta_0 - 1)$$

$$\frac{d|\lambda|^2}{dA}|_{\theta_0,A^*} = \frac{-8(\beta+1)(2\cos^2\theta_0 - 1) + 6\beta\cos\theta_0 - 4(\beta-A^*+1)}{L}$$

$$= \frac{-16(\beta+1)\cos^2\theta_0 + 6\beta\cos\theta_0 + 8(\beta+1) - 4(\beta - A^* + 1)}{L}$$

$$= \frac{-16(\beta+1)\cos^2\theta_0 + 6\beta\cos\theta_0 + 4(\beta + A^* + 1)}{L}$$

Suppose that $d|\lambda^2|/dA|_{\theta_0,A^*} = 0$, and substitute

$$\beta + A^* + 1 = \beta + 1 + \frac{4(\beta+1)}{(\beta+2)^2} = (\beta+1)(1 + \frac{4}{(\beta+2)^2})$$

$$= \frac{(\beta+1)}{(\beta+2)^2}\Big((\beta+2)^2 + 4)\Big)$$

$$\frac{-16\beta^2(\beta+1)}{4(\beta+2)^2} + \frac{6\beta^2}{2(\beta+2)} + \frac{(\beta+1)}{(\beta+2)^2}\Big((\beta+2)^2 + 4)\Big) = 0$$

$$\frac{-4\beta^2(\beta+1)}{(\beta+2)^2} + \frac{(\beta+1)}{(\beta+2)^2}\Big((\beta+2)^2 + 4)\Big) + \frac{3\beta^2(\beta+2)}{(\beta+2)^2} = 0$$

$$\frac{\beta^2(-4\beta - 4 + 3\beta + 6)}{(\beta+2)^2} + \frac{(\beta+1)}{(\beta+2)^2}\Big((\beta+2)^2 + 4)\Big) = 0$$

$$\frac{\beta^2(2-\beta) + (\beta+1)(\beta^2 + 4\beta + 8)}{(\beta+2)^2} = \frac{7\beta^2 + 12\beta + 8}{(\beta+2)^2} > 0$$

A contradiction so $d|\lambda^2|/dA|_{\theta_0,A^*} \neq 0$ □

We have shown that system (3) undergoes a Neimark-Sacker bifurcation. Now we determine the direction of stability of the invariant closed curve that bifurcates from the positive fixed point. We follow the normal form theory of Neimark-Sacker bifurcation as in [1]. Shift the fixed point to the origin by taking

$\begin{pmatrix} x_n \\ y_n \\ z_n \\ w_n \end{pmatrix} = \begin{pmatrix} U_n - U^* \\ V_n - V^* \\ W_n - W^* \\ Z_n - Z^* \end{pmatrix}$ then equation (5.2.2) becomes,

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\beta(x_n+X^*)+w_n+X^*}{A+y_n+X^*} - X^* \\ x_n \\ y_n \\ z_n \end{pmatrix} \tag{5.3.13}$$

Which can be written as

$$Y_{n+1} = JY_n + G(Y_n) \tag{5.3.14}$$

where $G(Y) = \frac{1}{2}B(Y,Y)+\frac{1}{6}C(Y,Y,Y)+O(||Y||^4)$ and $Y_n = \begin{pmatrix} x_n \\ y_n \\ z_n \\ w_n \end{pmatrix}$

$B(Y,Y) = \begin{pmatrix} B_1(Y,Y) \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $C(Y,Y,Y) = \begin{pmatrix} C_1(Y,Y,Y) \\ 0 \\ 0 \\ 0 \end{pmatrix}$

where

$$B_i(x,y) = \sum_{j,k=1}^{n} \frac{\partial^2 Y_i(\xi)}{\partial\xi_j \partial\xi_k}|_{\xi=0}(x_j y_k)$$

and

$$C_i(x,y,z) = \sum_{j,k,l=1}^{n} \frac{\partial^3 Y_i(\xi)}{\partial\xi_j \partial\xi_k \partial\xi_l}|_{\xi=0}(x_j y_k z_l)$$

$$B_1(\phi,\psi) = \frac{-\beta}{(\beta+1)^2}(\phi_2\psi_1+\phi_1\psi_2)+2\frac{\beta-A+1}{(\beta+1)^2}\phi_2\psi_2+\frac{-1}{(\beta+1)^2}(\phi_2\psi_4+\phi_4\psi_2)$$

$$C_1(\phi,\psi,\eta) = \frac{-6(\beta-A+1)}{(\beta+1)^3}\phi_2\psi_2\eta_2 + \frac{2\beta}{(\beta+1)^3}(\phi_2\psi_2\eta_1+\phi_1\psi_2\eta_2+\phi_2\psi_1\eta_2)$$
$$+\frac{2}{(\beta+1)^3}(\phi_2\psi_2\eta_4+\phi_4\psi_2\eta_2+\phi_2\psi_4\eta_2)$$

Let $Jq^* = e^{i\theta_0}q^*$, $J^T p = e^{-i\theta_0}p$ where q^* and p are the eigenvectors corresponding to the eigenvalues $e^{i\theta_0}$ and $e^{-i\theta_0}$, respectively .

Solving $(J - \lambda I)q^* = (J - e^{i\theta_0}I)q^* = 0$

$$\begin{pmatrix} \frac{\beta}{\beta+1} - e^{i\theta_0} & \frac{-(\beta-A+1)}{\beta+1} & 0 & \frac{1}{\beta+1} \\ 1 & -e^{i\theta_0} & 0 & 0 \\ 0 & 1 & -e^{i\theta_0} & 0 \\ 0 & 0 & 1 & -e^{i\theta_0} \end{pmatrix} \begin{pmatrix} q_1^* \\ q_2^* \\ q_3^* \\ q_4^* \end{pmatrix} = 0$$

Let $q_1^* = 1$, from the second equation

$$1 - e^{i\theta}q_2^* = 0, \quad \text{so } q_2^* = e^{-i\theta_0}$$

From the third equation, $q_2^* - e^{i\theta_0}q_3^* = 0$, then

$$e^{-i\theta_0} = e^{i\theta_0}q_3^*, \quad \text{and } q_3^* = e^{-2i\theta_0}$$

From the fourth equation $q_3^* - e^{i\theta_0}q_4^* = 0$ then

$$e^{-2i\theta_0} = e^{i\theta_0}q_4^*, \quad \text{and } q_4^* = e^{-3i\theta_0}$$

We obtain $q^* \sim \begin{pmatrix} 1 \\ e^{-i\theta_0} \\ e^{-2i\theta_0} \\ e^{-3i\theta_0} \end{pmatrix}$

Note that this choice of q^* satisfies the first equation too. To have a non-zero solution of the system $(J - \lambda I)q^* = 0$, the matrix $(J - \lambda I)$ must be singular, that means $|J - \lambda I| = 0$.

$$|J - \lambda I| = \left(\frac{\beta}{\beta+1} - e^{i\theta_0}\right)(-e^{3i\theta_0}) + \frac{\beta - A + 1}{\beta+1}(e^{2i\theta_0}) - \frac{1}{\beta+1} = 0$$

For the first equation

$$\frac{\beta}{\beta+1} - e^{i\theta_0} - \frac{\beta - A + 1}{\beta+1}e^{-i\theta_0} + \frac{1}{\beta+1}e^{-3i\theta_0} = 0$$

Multiply by $-e^{3i\theta_0}$, then

$$\left(\frac{\beta}{\beta+1} - e^{i\theta_0}\right)(-e^{3i\theta_0}) + \frac{\beta - A + 1}{\beta+1}(e^{2i\theta_0}) - \frac{1}{\beta+1} = 0$$

Also, solving $(J-\lambda I)^T p = (J-e^{-i\theta_0}I)^T p = 0$

$$\begin{pmatrix} \frac{\beta}{\beta+1}-e^{-i\theta_0} & 1 & 0 & 0 \\ -\frac{\beta-A+1}{\beta+1} & -e^{-i\theta_0} & 1 & 0 \\ 0 & 0 & -e^{-i\theta_0} & 1 \\ \frac{1}{\beta+1} & 0 & 0 & -e^{-i\theta_0} \end{pmatrix}\begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = 0$$

Let $p_1 = 1$, from the first equation

$$\tfrac{\beta}{\beta+1} - e^{-i\theta_0} + q_2 = 0, \text{ thus } p_2 = -\tfrac{\beta}{\beta+1} + e^{-i\theta_0}$$

From the third equation

$$-e^{-i\theta_0}p_3 + \tfrac{e^{i\theta_0}}{\beta+1} = 0, \text{ thus } p_3 = \tfrac{e^{2i\theta_0}}{\beta+1}$$

From the fourth equation

$$\tfrac{1}{\beta+1} - e^{-i\theta_0}p_4 = 0, \text{ therefore } p_4 = \tfrac{e^{i\theta_0}}{\beta+1}$$

Note that this choice of p satisfies the second equation too. We obtain

$$p \sim \begin{pmatrix} 1 \\ -\frac{\beta}{\beta+1} + e^{-i\theta_0} \\ \frac{e^{2i\theta_0}}{\beta+1} \\ \frac{e^{i\theta_0}}{\beta+1} \end{pmatrix}$$

To normalize p and q^*, we must have $< p, q^* >= 1$, where $< .,. >$ is the standard scalar product in $\mathbb{C}^3$.

$$\eta =< p, q^* >= \begin{pmatrix} 1 & e^{i\theta_0} - \frac{\beta}{\beta+1} & \frac{e^{-2i\theta_0}}{\beta+1} & \frac{e^{-i\theta_0}}{\beta+1} \end{pmatrix}\begin{pmatrix} 1 \\ e^{-i\theta_0} \\ e^{-2i\theta_0} \\ e^{-3i\theta_0} \end{pmatrix}$$

$$\eta = 1 + 1 - \frac{\beta}{\beta+1}e^{-i\theta_0} + \frac{e^{-4i\theta_0}}{\beta+1} + \frac{e^{-4i\theta_0}}{\beta+1}$$

$$\eta = 2 - \frac{\beta}{\beta+1}e^{-i\theta_0} + 2\frac{e^{-4i\theta_0}}{\beta+1}$$

So let $q = \eta^{-1}q^*$, where $\eta^{-1} = 1/\eta$

The critical real eigenspace T^c corresponding to $\lambda_{1,2}$ is two-dimensional and is spanned by $\{\mathbb{R}e(q), \mathbb{I}m(q)\}$. The real eigenspace T^s corresponding to the real eigenvalues of J is two-dimensional. Any vector $x \in \mathbb{R}^4$ may be decomposed as

$$x = zq + \bar{z}\bar{q} + y$$

where $z \in \mathbb{C}^1$, and $\bar{z}\bar{q} \in T^c$, $y \in T^s$. The complex variable z is a coordinate on T^c. We have

$$\begin{cases} z =< p, x > \\ y = x- < p, x > q- < \bar{p}, x > \bar{q} \end{cases}$$

In these coordinates, the map (5.3.14) takes the form

$$\begin{cases} \tilde{z} = e^{i\theta_0}z+ < p, G(zq + \bar{z}\bar{q} + y) > \\ \tilde{y} = Jy + G(zq + \bar{z}\bar{q} + y)- < p, G(zq + \bar{z}\bar{q} + y) > q- < \bar{p}, G(zq + \bar{z}\bar{q} + y) > \bar{q} \end{cases}$$

The previous system can be written in the form:

$$\begin{cases} \tilde{z} = e^{i\theta_0}z + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2 + \frac{1}{2}G_{21}z^2\bar{z}+ < G_{10}, y > z+ < G_{01}, y > \bar{z} \\ \tilde{y} = Jy + \frac{1}{2}H_{20}z^2 + H_{11}z\bar{z} + \frac{1}{2}H_{02}\bar{z}^2 + \frac{1}{2}H_{21}z^2\bar{z} \end{cases}$$

Where $\begin{cases} G_{20} =< p, B(q, q) >, G_{11} =< p, B(q, \bar{q}) >, \\ G_{02} =< p, B(\bar{q}, \bar{q}) >, G_{21} =< p, C(q, q, \bar{q}) > \end{cases}$

$$\begin{cases} H_{20} = B(q, q)- < p, B(q, q) > q- < \bar{p}, B(q, q) > \bar{q} \\ H_{11} = B(q, \bar{q})- < p, B(q, \bar{q}) > q- < \bar{p}, B(q, \bar{q}) > \bar{q} \end{cases}$$

$\{ < G_{10}, y >=< p, B(q, y) >, < G_{01}, y >=< p, B(\bar{q}, y) >$
And the scalar product in $\mathbb{C}^3$ is used.
From the center manifold theorem, there exists a center manifold W^c which can be approximated as

$$Y = V(z, \bar{z}) = \frac{1}{2}w_{20}z^2 + w_{11}z\bar{z} + \frac{1}{2}w_{02}\bar{z}^2$$

where $< q, w_{ij} >= 0$. The vectors $w_{ij} \in \mathbb{C}^3$ can be found from the linear equations

$$\begin{cases} w_{20} = (e^{2i\theta_0} I_3 - J)^{-1} H_{20} \\ w_{11} = (I_3 - J)^{-1} H_{11} \\ w_{02} = (e^{-2i\theta_0} I_3 - J)^{-1} H_{02} \end{cases}$$

So z can be expressed as

$$\widetilde{z} = e^{i\theta_0}\bar{z} + \frac{1}{2}G_{20}z^2 + G_{11}z\bar{z} + \frac{1}{2}G_{02}\bar{z}^2$$

$$+\frac{1}{2}(G_{21} + 2 < p, B(q, (I - J)^{-1}H_{11}) > + < p, B(\bar{q}, (e^{2i\theta_0}I - J)^{-1}H_{20}) >)z^2\bar{z}$$

Taking into account the identities

$$(I - J)^{-1}q = \frac{1}{1 - e^{i\theta_0}}q, \qquad (e^{2i\theta_0}I - J)^{-1}q = \frac{e^{-i\theta_0}}{e^{i\theta_0} - 1}q$$

and

$$(I - J)^{-1}\bar{q} = \frac{1}{1 - e^{i\theta_0}}\bar{q}, \qquad (e^{2i\theta_0}I - J)^{-1}\bar{q} = \frac{e^{-i\theta_0}}{e^{i\theta_0} - 1}\bar{q}$$

We can express z using the map

$$\widetilde{z} = e^{i\theta_0}z + \sum_{k+l\geq 2}\frac{1}{k!j!}g_{k_j}z^k\bar{z}^j$$

where

$$g_{20} =< p, B(q, q) >, \qquad g_{11} =< p, B(q, \bar{q}) >, \qquad g_{02} =< p, B(\bar{q}, \bar{q}) >$$

$g_{21} =< p, C(q, q, \bar{q}) > +2 < p, B(q, (I - J)^{-1}B(q, \bar{q})) > +$
$< p, B(\bar{q}, (e^{2i\theta_0}I - J)^{-1}B(q, q)) > + \cdots$ or equivalently

$$\widetilde{z} = e^{i\theta_0}z(1 + d(\beta^*))|z^2|$$

where the real number $\beta(A^*) = \mathbb{R}e(d(A^*))$ that determines the direction of bifurcation of a closed invariant curve, can be computed via

$$\beta(A^*) = \mathbb{R}e(\frac{e^{-i\theta_0}g_{21}}{2}) - \mathbb{R}e(\frac{(1 - 2e^{i\theta_0})e^{-2i\theta_0}}{2(1 - e^{i\theta_0})}g_{20}g_{11}) - \frac{1}{2}|g_{11}|^2 - \frac{1}{4}|g_{02}|^2$$

Theorem 5.3.4. *If $\beta(A^*) < 0$(respectively, > 0), then the Neimark- Sacker bifurcation at $A = A^*$ is supercritical(respectively, subcritical) and there exists a unique invariant closed curve that bifurcates from the fixed point which is asymptotically stable (respectively, unstable).*

6. Computer Simulation

To illustrate the analytical results found, let us consider the following particular cases of equation 4.2.1. Notice the birth of closed curve and its direction.

```
N=300; x(1)=50;x(2)=50; A=0.8; x(3)=50;B=0;
for B=0:0.001:0.2;
for n=3:1:0.3*N
x(n+1)=(B*x(n)+x(n-2))/(A+x(n-1));
end
figure(1), hold on
for n=0.3*N:1:N
x(n+1)=(B.*x(n)+x(n-2))/(A+x(n-1));
plot(B,x(n+1),'.','MarkerSize',4)
axis ([0 0.5 0 3])
xlabel('B'),ylabel('x(n+1)'), grid on
end
end
hold off
```

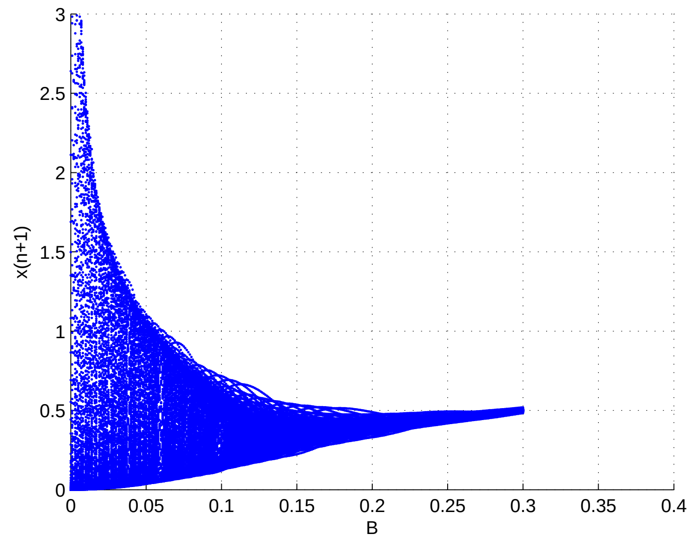

Fig. 6.1: Dynamical behavior

```
N=1000; x(1)=50;x(2)=50; B=0.33333; x(3)=50;A=0.5;
for A=0.5
for n=3:1:0.3*N
x(n+1)=(B.*x(n)+x(n-2))/(A+x(n-1));
x(n-2)
end
figure (2), hold on
for n=0.3*N:1:N
x(n+1)=(B.*x(n)+x(n-2))/(A+x(n-1))
x(n)
plot(x(n),x(n-2),'.','MarkerSize',5)
xlabel('x(n)'),ylabel('x(n-2)'),grid on
end
end
hold off
```

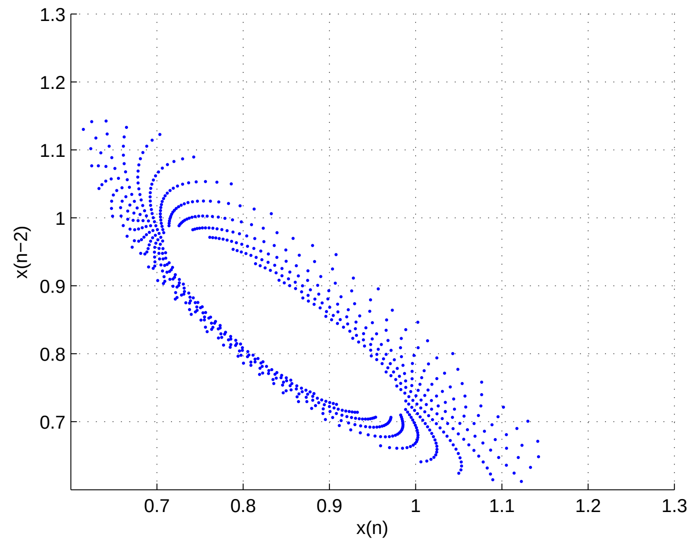

Fig. 6.2: Phase portrait at bifurcation value

```
N=1000; x(1)=50;x(2)=50; A=0.5; x(3)=50;B=0:0.001: 1;
for B=0.5;
for n=3:1:0.02*N
x(n+1)=(B.*x(n)+x(n-2))/(A+x(n-1));
x(n-2)
end
figure (3), hold on
for n=0.02*N:1:N
x(n+1)=(B.*x(n)+x(n-2))/(A+x(n-1));
x(n)
plot(x(n),x(n-2),'.','MarkerSize',5)
xlabel('x(n)'),ylabel('x(n-2)'),grid on
end
end
hold off
```

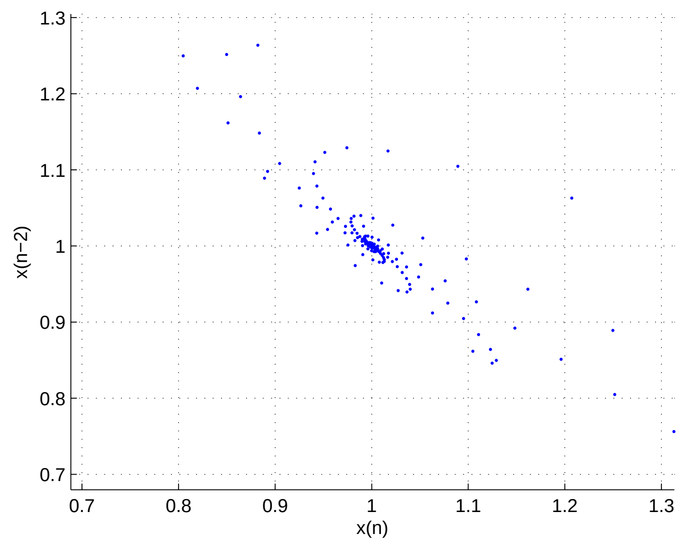

Fig. 6.3: Phase portrait away from bifurcation value

To illustrate the analytical results found for the fourth order rational difference equation 5.2.1. Let us consider the following particular cases of equation 5.2.1, and note the birth of closed curve and its direction.

```
N=300; x(1)=1;x(2)=1; B=1; x(3)=1; x(4)=1;
for A=0.8:0.001:1;
for n=4:1:0.2*N
x(n+1)=(B*x(n)+x(n-3))/(A+x(n-1))
end
figure(1), hold on
for n=0.2*N:1:N
x(n+1)=(B.*x(n)+x(n-3))/(A+x(n-1))
plot(A,x(n+1),'.','MarkerSize',5)
axis ([0.8 1 0 5])
xlabel('A'),ylabel('x(n+1)'),grid on
end
end
hold off
```

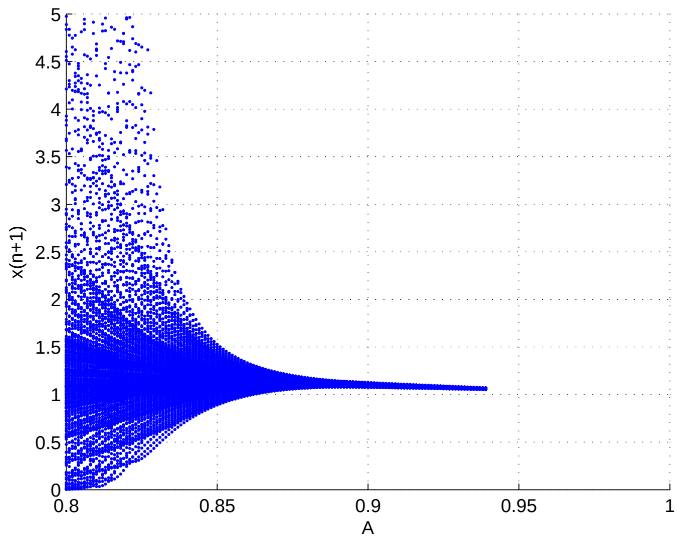

Fig. 6.4: Dynamical behavior

```
N=1000; x(1)=1.7;x(2)=1.7; B=1; x(3)=1.7;x(4)=1.7;
for A=0.8889
for n=4:1:0.3*N
x(n+1)=(B.*x(n)+x(n-3))/(A+x(n-1));
x(n-2)
end
figure (2), hold on
for n=0.3*N:1:N
x(n+1)=(B.*x(n)+x(n-3))/(A+x(n-1));
x(n)
plot(x(n),x(n-3),'.','MarkerSize',6)
xlabel('x(n)'),ylabel('x(n-3)'),grid on
end
end
hold off
```

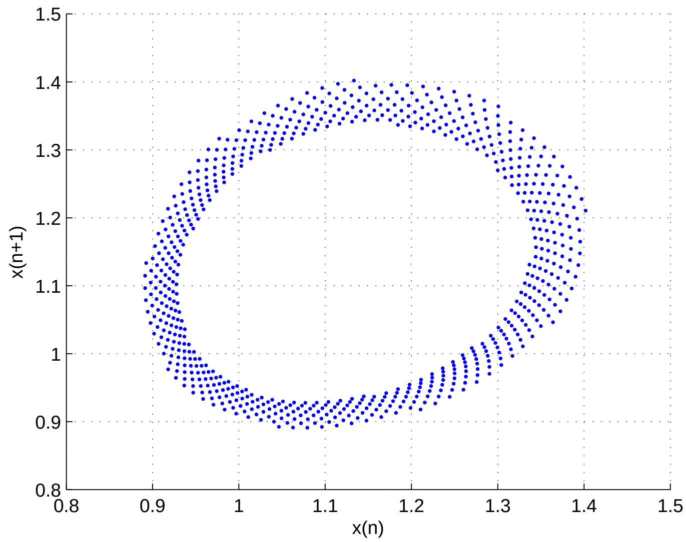

Fig. 6.5: Phase portrait at bifurcation value

```
clc
N=1000; x(1)=1;x(2)=1; B=0.9; x(3)=1;x(4)=1;
 for A=0.877
 for n=4:1:0.3*N
 x(n+1)=(B.*x(n)+x(n-3))/(A+x(n-1));
 x(n-2)
 end
 figure (3), hold on
 for n=0.3*N:1:N
 x(n+1)=(B.*x(n)+x(n-3))/(A+x(n-1));
 x(n)
 plot(x(n),x(n-3),'.','MarkerSize',6)
 xlabel('x(n)'),ylabel('x(n-3)'),grid on
  end
end
 hold off
```

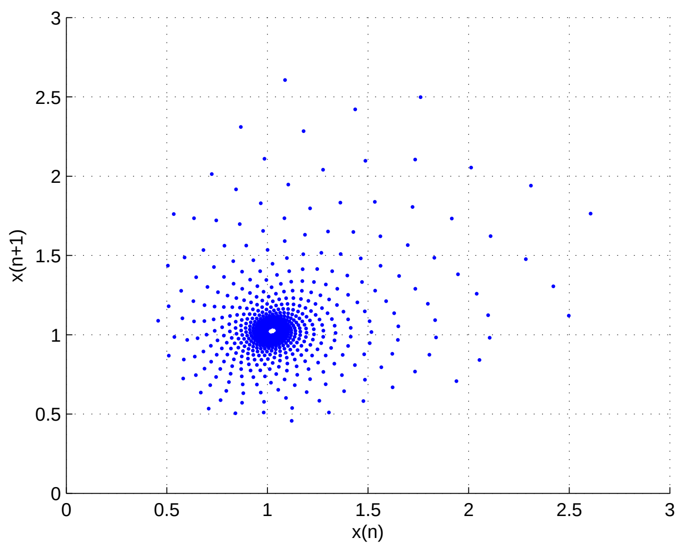

Fig. 6.6: Phase portrait away from bifurcation value

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